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MRC Technical Summary Report #2308

CLASSICAL SOLUTIONS OF THE KORTWEG-deVRIES EQUATION FOR NON-SMOOTH INITIAL DATA VIA INVERSE SCATTERING

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November 1981

Received September 1, 1981

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
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ABSTRACT

The Cauchy problem for the Korteweg-deVries equation (KdV for short)

$$(*)\qquad \begin{cases} q_t(x,t) + q_{xxx}(x,t) - 6q(x,t)q_x(x,t) = 0 \\ q(x,0) = Q(x) \end{cases}$$

is solved classically under the single assumption

$$\int_{-\infty}^{\infty} (1 + |x|)^4 |Q(x)| dx < \infty$$

for $t > 0$ via the so-called "inverse scattering method". This approach, originating with Gardner, Greene, Kruskal, and Miura [9], relates the KdV equation to the one-dimensional Schrödinger equation:

$$(**)\qquad -f''(x,k) + u(x)f(x,k) = k^2 f(x,k).$$

By considering the effect on the scattering data associated to the Schrödinger equation (**) when the potential $u(x)$ evolves in t according to the KdV equation (*), one obtains a linear evolution equation for the scattering data. The inverse scattering method of solving (*) consists of calculating the scattering data for the initial value $Q(x)$, letting it evolve to time t, and then recovering $q(x,t)$ from the evolved scattering data.

Recently, P. Deift and E. Trubowitz [7] presented a new method for solving the inverse scattering problem (obtaining the potential from its scattering data). Our solution of the KdV initial value problem uses this approach to construct a classical solution under the assumption stated above.

AMS(MOS) Subject Classification: 35C99, 35Q20, 34B25

Key Words: KdV equation, Cauchy problem, Inverse scattering

Work Unit No. 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

The Korteweg-deVries equation (KdV for short) arises as an approximation in many problems involving non-linear dispersive waves and has been extensively studied in recent years [1,3-6,9,11-13,15-19,21-28]. One approach to solving this equation in a more or less explicit fashion is the inverse scattering method of Gardner, Greene, Kruskal, and Miura [9], which relates the KdV equation to a one-parameter family of one-dimensional Schrödinger operators. We solve the Cauchy problem for the KdV equation by this method, using the inverse scattering theory of Deift and Trubowitz [7]. Previous authors [5,6,9,22] have done this using Faddeev's version of inverse scattering [8]; our approach constructs a classical solution under less restrictive conditions on the initial data. In particular, no smoothness is assumed. Some aspects of the asymptotic behavior of the solution are also discussed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CLASSICAL SOLUTIONS OF THE KORTEWEG-deVRIES EQUATION

FOR NON-SMOOTH INITIAL DATA VIA INVERSE SCATTERING

Robert L. Sachs

1. Introduction.

In this paper, we construct classical solutions of the Cauchy problem for the Korteweg-deVries (KdV for short) equation:

$$(*)\qquad \begin{cases} q_t + q_{xxx} - 6qq_x = 0, & t > 0, \quad x \in \mathbb{R} \\ q(x,0) = Q(x) \end{cases}$$

via the inverse scattering method of Gardner, Greene, Kruskal, Miura, and Zabusky (see [9,17]). This approach relies on the intimate connection between the KdV equation and the one-dimensional Schrödinger equation with potential $u(x)$:

$$(**)\qquad -f''(x,k) + u(x)f(x,k) = k^2 f(x,k).$$

More precisely, if one considers the one parameter family of Schrödinger operators (parameterized by t) whose potentials satisfy the KdV equation, then these operators are unitarily equivalent to one another. Moreover, by considering the so-called scattering data associated to these operators, one obtains a system of uncoupled, linear equations. The inverse scattering method of solving the KdV equation consists of first calculating initial values for the scattering data from the initial potential $Q(x)$, then solving the linear equations for the evolution of the scattering data, and finally, recovering the potential $q(x,t)$ at time t from its scattering data.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by he National Science Foundation under Grant No. MCS-7927062, Mod. 1.

The first two steps in this procedure are relatively straightforward; most of the technical difficulty in applying the method arises in the process of recovering the potential at time t. Previous authors, including Tanaka [22] and Cohen-Murray [5], used the inverse scattering theory of Faddeev [8] to complete this step under suitable hypotheses on the initial data (involving decay of several derivatives). In a later paper, Cohen-Murray [6] considered initial data which was continuous with a piecewise continuous first derivative. Assuming a certain decay rate, she proved the existence of a smooth solution. Our result does not assume a specific decay rate but rather requires that the initial data belong to a weighted L^1 space. We make no assumptions on derivatives of the initial data. In recovering the potential from its scattering data, we use the recent work of P. Deift and E. Trubowitz [7] on the inverse scattering problem. One difference between this theory and Faddeev's is that the so-called trace formula of [7] (see also Newell [19]) expresses the potential itself directly in terms of the scattering data. In this way, the solution $q(x,t)$ of the KdV equation (*) and the solution $h(x,t)$ of the linear problem:

$$(1.1) \quad \begin{cases} h_t + h_{xxx} = 0 \\ h(x,0) = H(x) \end{cases}$$

are compared directly, where $H(x)$ is related to the scattering data of $Q(x)$. Our analysis consists in large part of solving (1.1) in various weighted, local L^1 and L^∞ spaces and then using the link between $h(x,t)$ and $q(x,t)$ to extend these properties to $q(x,t)$.

Several other approaches to the KdV initial value problem have been developed besides the inverse scattering method used here. Saut and Temam [21] used a parabolic regularization of the KdV equation to establish the existence and uniqueness of solutions in the Hilbert spaces H^s for $s \geq 2$. Bona and Smith [4] obtained existence, uniqueness, and continuous dependence on initial data in H^s for all integers $s \geq 2$ using a regularizing term of order 3 (the so-called regularized long wavelength equation). Bona and Scott [3], using non-linear interpolation theory, extended this result to non-integer values of s. These results show that in the Hilbert spaces H^s, the KdV evolution is not smoothing in the strong sense but preserves the order of L^2-differentiability. Recently, Kato [12,13], using the abstract theory of quasi-linear evolution equations, proved the existence and weakly continuous dependence on initial data in H^s, $s = 0,1$, and the weighted Sobolev spaces

$$H^{2r,r} \equiv H^{2r} \cap L^2((1+x^2)^r dx)$$

and

$$H^s \cap L_b^2, \quad \text{where} \quad L_b^2 \equiv L^2(e^{2bx}dx)$$

for $b > 0$, $s \geq 0$.

Kato also shows that in fact, for $s > 3/2$, an H^s solution belongs to $L^2([0,T];H_{loc}^{s+1})$, which is a kind of smoothing effect. Other papers on the Cauchy problem for the KdV equation on the line include [11,18,15-27]. For a nice survey article, see Miura [17].

Our result, as well as that of Cohen-Murray [5,6], exhibits an interplay between smoothness and decay in the pointwise and weighted L^1 senses. We remark that these contrasts arise in the linear evolution equation (1.1) as well as for the KdV equation.

Before describing our results, we introduce some notation:

$$(1.2)\quad \begin{cases} L_r^p(a) := L_r^p([a,\infty)) \\ \qquad\quad = \{u(x) : (1+|x|)^r u(x) \in L^p([a,\infty))\} \\ X_r^p \quad := \bigcap\limits_{a \text{ finite}} L_r^p(a) \end{cases}$$

L_r^p will denote $L_r^p((-\infty,\infty))$. Note that X_r^p is a Fréchet space with the obvious topology.

Our principal results are as follows:

Suppose the initial value $Q(\cdot) \in L_\alpha^1$. Then there exists a unique solution $q(x,t)$ of (*) with the properties:

(i) $q(x,t) \in X^1_{\alpha-3/4-\delta} \cap X^\infty_\alpha$ for every $t > 0$ and $\delta > 0$ and $t \to q(\cdot,t)$ is continuous in this topology for $t > 0$.

(ii) $\partial_t^r \partial_x^s q(\cdot,t) \in X^1_{\alpha-3/4-\delta-\left(\frac{s+3r}{2}\right)} \cap X^\infty_{\alpha-\left(\frac{s+3r}{2}\right)}$

(with continuity in t)

(iii) $q(x,t) \to Q(x)$ as $t \downarrow 0$ in $X^1_{\alpha-1-\delta}$ for every $\delta > 0$.

As yet, we do not know how to prove that the solution constructed has the same asymptotic behavior as $x \to -\infty$ or as $t \to +\infty$ (modulo solitons) as the solution of the linear problem (1.1), although we believe this to be true (See [1]).

The basic results of inverse scattering theory, including sketches of the approaches of Faddeev [8] and Deift and Trubowitz [7], appear in Section 2 below. In Section 3, the link between the KdV equation and the Schrödinger equation is described and the evolution of the scattering data given. Using this evolution, the reconstruction of the potential by the Deift-Trubowitz method is discussed in Section 4. Additional smoothness and decay properties of the proposed solution are developed in Section 5, while Section 6 contains the proof that our solution indeed satisfies the KdV equation.

It is a pleasure to acknowledge helpful conversations with Percy Deift and Jerry Bona during the course of this work and the support of an A.M.S Postdoctoral Research Fellowship.

2. Results From Inverse Scattering Theory.

In this section, we sketch the results of the theory of inverse scattering for the one-dimensional Schrödinger equation [7,8] to be used in studying the KdV initial value problem. Where feasible, the proofs of these results are described briefly. Our discussion is divided into three parts: the definition of scattering data for the Schrödinger equation; Faddeev's inverse method [8]; and the Deift-Trubowitz approach [7]. Our discussion is for the reader's benefit and contains no new results.

A. The Forward Problem: Defining the Scattering Data; Basic Properties.

Consider the Schrödinger equation

$$(**) \qquad -f''(x,k) + u(x)f(x,k) = k^2 f(x,k)$$

on the real line $-\infty < x < +\infty$ for k real and non-zero. Assuming that the potential $u(x)$ is real-valued and satisfies the condition:

$$(2.1) \qquad \|u\|_{L_1^1} \equiv \int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty$$

we construct the Jost solutions $f_\pm(x,k)$ of (**) as follows:

Solve the Volterra integral equations:

$$(2.2) \qquad \begin{cases} m_+(x,k) = 1 + \displaystyle\int_x^{\infty} \left(\frac{e^{2ik(y-x)}-1}{2ik}\right) u(y)m_+(y,k)dy \\[2ex] m_-(x,k) = 1 + \displaystyle\int_{-\infty}^{x} \left(\frac{e^{-2ik(y-x)}-1}{2ik}\right) u(y)m_-(y,k)dy \end{cases}$$

for $m_\pm(x,k)$ and then define

$$(2.3) \qquad f_\pm(x,k) = e^{\pm ikx} m_\pm(x,k).$$

Condition (2.1) arises when estimating the terms in parentheses by $|y-x|$, obtaining a bound independent of k. Then $f_\pm(x,k)$ satisfy the Schrödinger equation (**) and have the asymptotic behavior

$$(2.4)\quad \begin{cases} f_+(x,k) \sim e^{ikx} & \text{as} \quad x \to +\infty \\ f_-(x,k) \sim e^{-ikx} & \text{as} \quad x \to -\infty. \end{cases}$$

Since $u(x)$ and k are real, $\overline{f_\pm(x,k)} = f_\pm(x,-k)$ are also solutions of (**). (2.4) implies $f_+(x,k)$ and $f_+(x,-k)$ are linearly independent for $k \neq 0$ (and similarly for f_-) since

$$[f_+(x,k),f_+(x,-k)] = [f_-(x,-k),f_-(x,k)] = 2ik.$$

Thus there are functions $T_\pm(k),R_\pm(k)$ defined for real, non-zero k by the relations:

$$(2.5)\quad \begin{cases} f_-(x,k) = \dfrac{1}{T_+(k)} f_+(x,-k) + \dfrac{R_+(k)}{T_+(k)} f_+(x,k) \\[2ex] f_+(x,k) = \dfrac{1}{T_-(k)} f_-(x,-k) + \dfrac{R_-(k)}{T_-(k)} f_-(x,k). \end{cases}$$

Taking Wronskians as above, we obtain:

$$(2.6)\quad \begin{cases} \dfrac{2ik}{T_+(k)} = \dfrac{2ik}{T_-(k)} = [f_+(x,k),f_-(x,k)] \\[2ex] \dfrac{2ikR_+(k)}{T_+(k)} = [f_-(x,k),f_+(x,-k)] \\[2ex] \dfrac{2ikR_-(k)}{T_-(k)} = [f_-(x,-k),f_+(x,k)] \end{cases}$$

$T_\pm(k)$ is called the transmission coefficient; $R_\pm(k)$ are the reflection coefficients. The scattering matrix, $S(k)$, is given as:

$$(2.7)\qquad S(k) = \begin{pmatrix} T_+(k) & R_-(k) \\ R_+(k) & T_-(k) \end{pmatrix}$$

and is well-defined for real $k \neq 0$.

The properties of the scattering matrix are summarized in the following:

Lemma 2.1. (c.f. Theorem 2.1 of [7]) The scattering matrix, $S(k)$, has the following properties:

(i) $S(k)$ is continuous for all real $k \neq 0$. Moreover, if

$$\|u\|_{L_2^1} \equiv \int_{-\infty}^{\infty} (1 + |x|)^2 |u(x)| dx < \infty$$

then $S(k)$ is also continuous at $k = 0$.

(ii) $T_+(k) = T_-(k) = T(k)$

(iii) $\overline{T(k)} = T(-k)$; $\overline{R_\pm(k)} = R_\pm(-k)$

(iv) $T(k)\overline{R_-(k)} + R_+(k)\overline{T(k)} = 0$;

$$|T(k)|^2 + |R_+(k)|^2 = |T(k)|^2 + |R_-(k)|^2$$
$$= 1.$$

Thus $|T(k)|$, $|R_\pm(k)| \leq 1$.

Moreover, $S(k)$ is unitary.

(v) $T(k)$ extends meromorphically in the upper half-plane $\operatorname{Im} k > 0$. $T(k)$ has a finite number of simple poles at $k = i\beta_1, \cdots, i\beta_n$ where each β_j is a real, positive number; the residue of $T(k)$ at $k = i\beta_j$ is

$$i\left[\int_{-\infty}^{\infty} f_+(x,i\beta_j) f_-(x,i\beta_j) dx\right]^{-1}.$$

$-\beta_1^2, -\beta_2^2, \cdots, -\beta_n^2$ are the eigenvalues (bound state energies) of the Schrödinger operator. In $\operatorname{Im} k \geq 0$, $T(k)$ is continuous away from $0, i\beta_1, \cdots, i\beta_n$ (if $u \in L_2^1$, then $T(k)$ is continuous at $k = 0$ as well).

(vi) $T(k) = 1 + O(1/k)$ as $|k| \to \infty$, $\operatorname{Im} k \geq 0$.

$R_{\pm}(k) = o(1/k)$ as $|k| \to \infty$, k real.

Moreover, if $u(x)$ has ℓ derivatives in L^1,

$R_{\pm}(k) = o\left(\frac{1}{k^{\ell+1}}\right)$ as $|k| \to \infty$, k real.

Also, if there are no eigenvalues,

$T(k) - 1 \in H^{2^+}$ (Hardy space)

and $|T(k)| \leq 1$ for all $\operatorname{Im} k \geq 0$.

(vii) $|T(k)| > 0$ for all $k \neq 0$, $\operatorname{Im} k \geq 0$.

$|k| \leq C|T(k)|$ as $k \to 0$;

If $u(x) \in L_2^1$ then <u>either:</u>

(a) $0 \leq C_1 \leq |T(k)|$; $|R_{\pm}(k)| \leq C_2 < 1$

or

(b) $T(k) = \alpha k + o(k)$ as $k \to 0$, $\operatorname{Im} k \geq 0$;

$R_{\pm}(k) = -1 + \gamma_{\pm} k + o(k)$ as $k \to 0$, k real.

Most of these properties are direct consequences of (2.5), (2.6) and the relation $\overline{f_{\pm}(x,k)} = f_{\pm}(x,-k)$. The remainder come from careful analysis and the integral representations (derived from (2.2) and (2.5)):

$$
(2.8) \quad \begin{cases}
\dfrac{2ikR_+(k)}{T(k)} = \displaystyle\int_{-\infty}^{\infty} e^{-2iky} u(y) m_-(y,k)\,dy \\[2ex]
\dfrac{2ikR_-(k)}{T(k)} = \displaystyle\int_{-\infty}^{\infty} e^{2iky} u(y) m_+(y,k)\,dy \\[2ex]
\dfrac{1}{T(k)} = 1 - \dfrac{1}{2ik} \displaystyle\int_{-\infty}^{\infty} u(y) m_{\pm}(y,k)\,dy.
\end{cases}
$$

Note that the functions $\dfrac{2ikR_{\pm}(k)}{T(k)}$ look like Fourier transforms of the potential $u(v)$, aside from the factors $m_{\pm}(y,k)$. For a full proof of Lemma 2.1, see [7].

In [7], it is also proved that $m_{\pm}(x,k) - 1$, considered as functions of k, belong to the Hardy space H^{2^+}. The Fourier transform of $m_{\pm}(x,k) - 1$ with respect to k plays an important role in inverse scattering theory.

As remarked in [8], the reflection coefficient $R_+(k)$ (or $R_-(k)$) and the values $\beta_1,\cdots,\beta_n$ are sufficient to uniquely determine the entire scattering matrix $S(k)$. Namely, since $|T(k)|^2 = 1 - |R_+(k)|^2$ for k real, given $R_+(k)$ we know $|T(k)|$ for k real. Multiplying by the product $\prod_{j=1}^{n} \frac{k-i\beta_j}{k+i\beta_j}$ to remove poles, we recover $T(k) \cdot \prod_{j=1}^{n} \frac{k-i\beta_j}{k+i\beta_j}$ by exponentiating the Cauchy integral for its logarithm. Then $R_-(k)$ is obtained via property (iv) in Lemma 2.1 above.

Given $R_+(k)$ and $\beta_1,\cdots,\beta_n$, does this determine the potential $u(x)$? It turns out that n additional pieces of information are needed. Typically, one specifies the so-called <u>norming constants,</u> c_j, <u>defined by</u>

$$c_j^{-1} \equiv \int_{-\infty}^{\infty} f_+^2(x,i\beta_j)\,dx. \tag{2.9}$$

Given $\{R_+(k) | k \in \mathbb{R}\}$, $\beta_1,\cdots,\beta_n$, and $c_1,\cdots,c_n$, which we call the <u>scattering data,</u> the potential $u(x)$ is unique (Levinson's Theorem). The basic goal of inverse scattering theory is to describe how one obtains the potential from its scattering data.

B. Faddeev's Inverse Scattering Theory.

We describe briefly one method of constructing the potential from its scattering data. This approach, due to Faddeev [8], is based on work of Gel'fand and Levitan [10], Kay and Moses [14], and Agranovich and Marchenko [2]. A linear integral equation for $B_{\pm}(x,y)$, the Fourier transform of $m_{\pm}(x,k) - 1$, is solved and $u(x)$ obtained by the relation:

$$u(x) = -\frac{\partial}{\partial x} B_{+}(x,0^{+}) = \frac{\partial}{\partial x} B_{-}(x,0^{-}). \tag{2.10}$$

More precisely, define

$$\Omega_{+}(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{+}(k)e^{2iky}dk + 2\sum_{j=1}^{n} c_j e^{-2\beta_j y}. \tag{2.11}$$

The integral equation for $B_{+}(x,y)$, $y > 0$, (often called the Gel'fand-Levitan-Marchenko equation) is:

$$B_{+}(x,y) + \Omega_{+}(x+y) + \int_{0}^{\infty}\Omega_{+}(x+y+z)B_{+}(x,z)dz = 0. \tag{2.12}$$

A similar equation for $B_{-}(x,y)$, $y < 0$, holds where

$$\begin{aligned} \Omega_{-}(y) &\equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R_{-}(k)e^{-2iky}dy + 2\sum_{j=1}^{n} \tilde{c}_j\, e^{2\beta_j y}; \\ \tilde{c}_j^{-1} &\equiv \int_{-\infty}^{\infty} f_{-}^{2}(x,i\beta_j)dx. \end{aligned} \tag{2.13}$$

We remark that (2.10) follows from the Fourier transform version of (2.2), while (2.12) is the Fourier transform of (2.5) when suitably expressed.

Since we will not use the Faddeev approach to the inverse scattering problem, we do not discuss the solvability of the Gel'fand-Levitan-Marchenko equation (2.12) or the properties of the potential obtained by this method. The interested reader is referred to Faddeev [8] and Deift and Trubowitz [7] for more information on this theory, and to the works of Gardner, Greene, Kruskal, and Miura [9], Tanaka [22], and Cohen-Murray [5,6] for its application to the KdV initial value problem. We merely note that recovery of the potential $u(x)$ via (2.10) does not provide a simple means of obtaining pointwise estimates for $u(x)$ in terms of $R_+(k)$, as it involves finding $B_+(x,y)$ and differentiating.

C. The Deift-Trubowitz Inverse Scattering Theory.

Recently, P. Deift and E. Trubowitz [7] presented a rather different approach to the inverse scattering problem. The key to their method is the following 'trace formula' (see also Newell [19])

$$u(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} k\, R_+(k)e^{2ikx}m_+^2(x,k)dk - 4\sum_{j=1}^{n} c_j\beta_j f_+^2(x,i\beta_j). \tag{2.14}$$

Note that (2.14) expresses the potential $u(x)$ directly in terms of the squares of the Jost functions, so that the Schrödinger equation may be thought of as a coupled system of second-order ordinary differential equations with the singular boundary value $m_+(x,k) \sim 1$ as $x \to +\infty$ given. This is the basic idea in the Deift-Trubowitz approach. Carrying out this program involves a number of technical questions, which we do not discuss in any detail here. However, for the reader's benefit and as an orientation for Section 5 below, we indicate the principal issues.

First of all, it is technically convenient to restrict attention to potentials without bound states. In Section 3 of [7], a spectral version of Crum's algorithm is used to add or subtract bound states. Thus the problem of recovering potentials with bound states from the scattering data is in principle reducible to the case of potentials without bound states. The method of reduction is based on commutation ofthe operator A^*A where A is closed. In particular, suppose $Q(x)$ is a potential with bound state Schrödinger eigenvalues $-\beta_n^2 < -\beta_{n-1}^2 < \cdots < -\beta_1^2$. For $-\beta^2 < -\beta_n^2$ let $g(x)$ be a positive solution of

$$\left(-\frac{d^2}{dx^2} + Q(x) + \beta^2\right)g(x) = 0. \tag{2.15}$$

Then it is not hard to verify that, if we define the closed operator A as

(2.16) $$Af = (g \frac{d}{dx} g^{-1})f$$

then $$A^*A = -\frac{d^2}{dx^2} + Q(x) + \beta^2$$

and $$AA^* = -\frac{d^2}{dx^2} + Q(x) - 2\frac{d^2}{dx^2} \log g(x) + \beta^2$$

$$= -\frac{d^2}{dx^2} + P(x) + \beta^2.$$

In [7] it is shown that A^*A and AA^* have the same spectrum except perhaps for 0. But $-\beta^2$, which was not an eigenvalue for the Schrödinger operator with potential $Q(x)$, is an eigenvalue for the same operator with potential $P(x) \equiv Q(x) - 2\frac{d^2}{dx^2} \log g(x)$ with $\frac{1}{g(x)}$ as eigenfunction. This is easily seen since $AA^*(\frac{1}{g}) = (-g \frac{d}{dx} g^{-2} \frac{d}{dx} g) \frac{1}{g} = 0$. Thus $P(x)$ has an extra bound state eigenvalue $-\beta^2$. This describes how one adds a bound state; reversing the procedure will remove one. In this process, the scattering data and eigenfunctions are transformed in a nice way. With $Q(\cdot)$, $P(\cdot)$ as above, the transmission coefficients, reflection coefficients, norming constants and eigenfunctions are related as follows:

(2.17) $$\begin{cases} T_P(k) = \frac{k+i\beta}{k-i\beta} T_Q(k) \\ R_P(k) = -\frac{k+i\beta}{k-i\beta} R_Q(k) \\ f_{+,P}(x,i\beta_j) = -\frac{A \cdot f_{+,Q}(x,i\beta_j)}{\beta_j+\beta}, \quad j = 1,\cdots,n \\ C_{j,P} = \frac{\beta+\beta_j}{\beta-\beta_j} C_{j,Q} \\ C_{n+1,P} = 2\left(\frac{\beta}{\alpha}\right) T_Q(i\beta) \\ \text{where we choose } g(x) \text{ above as} \\ g \equiv f_+(x,i\beta) + \alpha f_-(x,i\beta) \text{ with } \alpha > 0. \end{cases}$$

Thus any $-\beta^2 < -\beta_n^2$ and any $C_{n+1} > 0$ are attainable this way. Moreover, if $Q(\cdot) \in L_\mu^1$ so is $P(\cdot)$ for all $\mu \geq 1$ and the associated reflection and transmission coefficients have identical smoothness and decay properties. In [7], it is also shown that n bound states may be added or removed in one algebraic procedure. Thus there is no loss of generality in considering potentials without bound states in the Deift-Trubowitz scheme.

Now consider the Schrödinger equation

$$(2.18) \quad \begin{cases} m_+''(x,k) + 2ikm_+'(x,k) = Q(m_+;R)m_+(x,k) \\ m_+(x,k) \sim 1 \quad \text{as} \quad x \to +\infty \end{cases}$$

where $Q(m_+;R)$ is the right-hand side of (2.14) (with no bound states). Let $n_+(x,k) \equiv e^{2ikx}m_+'(x,k)$.

We have the equivalent system:

$$(2.19) \quad \begin{cases} m_+'(x,k) = e^{-2ikx}n_+(x,k) \\ n_+'(x,k) = Q(m_+;R)e^{2ikx}m_+(x,k) \\ \left.\begin{array}{l} m_+ \sim 1 \\ n_+ \sim 0 \end{array}\right\} \quad \text{as} \quad x \to +\infty. \end{cases}$$

In order to construct solutions to (2.19) on the Banach space $B = \left\{(m,n) \,\middle|\, \sup_{k \in \mathbf{R}} (|m| + |n|) < \infty\right\}$ a Lipschitz estimate on the vector field in (2.19) is established. Then a solution on semi-infinite interval $[M,\infty)$ is constructed, for M large, by a contraction mapping argument. Here the following condition on $R_+(k)$ is used:

(2.20) The function $F_+(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_+(k)e^{2iky}dy$

is absolutely continuous with:

$$\int_X^\infty |F_+'(y)|(1 + |y|^2))dy \leq c(X) < \infty \quad \text{for all} \quad -\infty < X < \infty.$$

We remark that this condition is rather natural when working with L_2^1 potentials, since $F_+'(y)$ and $u(y)$ are quite similar (replace m_+ by 1 and ignore bound states).

One of the main results of [7] is a sharp characterization theorem for potentials in L_2^1. In this class, necessary and sufficient conditions for a given set of data to be the scattering data of an L_2^1 potential are given by the following (Theorem 5.3 of [7]):

<u>Theorem 2.2.</u> A matrix $S(k) = \begin{pmatrix} S_{11}(k) & S_{12}(k) \\ S_{21}(k) & S_{22}(k) \end{pmatrix}$ is a scattering matrix a real potential $u(x) \in L_2^1$ <u>without bound states</u> if and only if conditions (i)-(vii) of Lemma 2.1 hold and <u>moreover</u> (2.20) and its analogue for $R_-(k)$ (as in (2.13) above) hold.

While this theorem seems to suggest a nice class of potentials for our problem, the dispersive nature of the KdV equation leads to poor decay of solutions as $x \to -\infty$ for $t > 0$ [1,6]. Therefore, as in Cohen-Murray [6], we will reconstruct the solution for $t > 0$ from $x = +\infty$ to the left, using the method sketched above. In this way, the estimate (2.17) is sufficient and no condition on $R_-(k)$ is needed. Our potential will be constructed for every finite x in this manner. One inherent difficulty is that the "left" eigenfunction $f_-(x,k,t)$ may not be constructed directly for $t > 0$ because of the poor decay of the potential as $x \to -\infty$. Thus the algebraic procedure for adding or removing bound states is no longer available and we are forced to include bound state contributions in the Deift-Trubowitz scheme. As we shall see, this is not a serious problem.

3. The Inverse Scattering Method for Solving the KdV Equation: Evolution of the Scattering Data.

In a series of papers culminating in [9], Gardner, Greene, Kruskal, and Miura studied the KdV equation

$$(*) \qquad q_t + q_{xxx} - 6qq_x = 0.$$

They discovered a remarkable link between this non-linear evolution equation and the one-parameter family of Schrödinger operators

$$(3.1) \qquad L(t) = -\frac{d^2}{dx^2} + q(x,t)$$

where $q(x,t)$, the 'potential' in $L(t)$, satisfies the KdV equation. If we consider the scattering data associated to $L(t)$, namely

$$(3.2) \qquad \mathcal{S}(t) \equiv \{R_+(k,t): k \in \mathbb{R}\} \cup \{\beta_j(t), c_j(t)\}_{j=1,\cdots,n}$$

then the time evolution of the scattering data, assuming a priori that it is well-defined, is linear and given by the following well-known formulae:

Lemma 3.1. If $q(x,t)$ evolves according to the KdV equation (*), the scattering data $\mathcal{S}(t)$ given in 3.2 satisfy the equations:

$$(3.3) \qquad \begin{cases} \text{(i)} & \frac{d}{dt} R_+(k,t) = 8ik^3 R_+(k,t) \\ \text{(ii)} & \frac{d}{dt} \beta_j(t) = 0 \\ \text{(iii)} & \frac{d}{dt} c_j(t) = 8\beta_j^3 c_j(t). \end{cases}$$

Remark. From the discussion of the previous section concerning the determination of $T(k)$ and $R_-(k)$ given $R_+(k)$ and $\beta_1,\cdots,\beta_n$, it follows that

$$\frac{d}{dt} T(k,t) = 0 \text{ and } \frac{d}{dt} R_-(k,t) = -8ik^3 R_-(k,t).$$

We present a proof of Lemma 3.1 due to Tanaka [22], based on the operator formalism introduced by Lax [16], which is slightly different from the original derivation in [9].

As noted by Lax [10], the KdV equation is equivalent to the operator equation:

$$\frac{d}{dt} L(t) = [B,L] = BL - LB \tag{3.4}$$

where the operator B is defined as follows $(D \equiv \frac{d}{dx})$:

$$B = -4D^3 + 3qD + 3Dq. \tag{3.5}$$

In the literature, the skew-adjoint operator B is often called the Lenard operator (after A. Lenard) and equation (3.4) is often referred to as a Lax pairing. Since B is formally skew-adjoint, it generates a unitary group, so (3.4) implies that $L(t)$ and $L(0)$ are unitarily equivalent for every t.

Considering the family of eigenvalue problems:

$$L(t)f(x,k,t) = k^2 f(x,k,t) \tag{3.6}$$

and differentiating with respect to t, we find

$$L(t)[f_t - Bf] = k^2[f_t - Bf]. \tag{3.7}$$

If $f(x,k,t)$ is a Jost solution $f_\pm(x,k,t)$, where $f_+(x,k,t) \sim e^{ikx}$ as $x \to +\infty$ for every t fixed and $f_-(x,k,t) \sim e^{-ikx}$ as $x \to -\infty$ for every t fixed, then, analyzing the asymptotic behavior of $f_t - Bf$, we find:

$$\begin{cases} (f_+)_t - Bf_+ = -4ik^3 f_+ \\ \\ (f_-)_t - Bf_- = 4ik^3 f_-. \end{cases} \tag{3.8}$$

Here we use the assumption $q(x,t)$, $q_x(x,t) \to 0$ as $|x| \to \infty$, t fixed. Differentiating the relation (2.5) with respect to t and using (3.8), we obtain (3.3)(i). (Alternatively, one can differentiate the Wronskians in (2.6) and obtain this result.) By our previous remark, $L(t)$ is unitarily equivalent to $L(0)$, thus $\beta_j(t) = \beta_j(0)$. To obtain (3.3)(iii), we differentiate $f_+(x,i\beta_j,t) = \alpha_j(t)f_-(x,i\beta_j,t)$ and use (3.8).

An interpretation of the linear flow (3.3) in the context of completely integrable Hamiltonian systems was given by Zakharov and Faddeev [28]. Introducing a Hamiltonian structure, they show that the variables

$$(3.9)\qquad \begin{cases} P(k) = -\frac{k}{\pi}\,\ell n(1 - |R_-(k)|^2) \\ \\ Q(k) = \arg\left(\frac{R_-(k)}{T(k)}\right) \\ \\ P_j = \beta_j^2 \\ \\ q_j = 2\ell n\left(i\tilde{c}_j \frac{d}{dk}\left(\frac{1}{T(k)}\right)\Big|_{k=i\beta_j}\right) \end{cases}$$

form action-angle variables for the KdV Hamiltonian. In fact, they derive the *expression*

$$(3.10)\qquad H[u] = 8\int_{-\infty}^{\infty} k^3 P(k)dk - \frac{32}{5}\sum_{\ell=1}^{n} P_\ell^{5/2}$$

which is equivalent to (3.3) reexpressed in terms of $R_-(k)$, $\tilde{c}_j$, β_j (the alternative set of scattering data using normalizations at $x = -\infty$).

4. Recovering the Potential at Later Times.

Shifting our viewpoint somewhat, we now consider the linear flow on the scattering data induced by the nonlinear KdV equation. We show that under suitable restrictions, the evolved scattering data is sufficiently "well-behaved" to permit recovery of an associated potential by the inverse scattering methods of Section 2 above. To this end, we make the following definition:

(4.1) A function $q(x,t)$ is called <u>a generalized solution of the KdV equation (in the sense of inverse scattering theory)</u> for all $t > 0$ if it is the potential corresponding to the scattering data:

(4.2) $$\mathcal{S}(t) \equiv \{R_+(k,0)e^{8ik^3t} \mid k \in \mathbb{R}\} \cup \{\beta_j, c_j(0)e^{8\beta_j^3 t} \mid j = 1,\cdots,n\}$$

for every $t > 0$.

To avoid confusion, the term "weak solution" will be used for solutions in the sense of distributions and "generalized solution" will always refer to the notion (4.1). The chief result of this section is the following theorem.

<u>Theorem 4.1.</u> Given $Q(x) \in L^1_4$, there exists a unique generalized solution (in the sense of inverse scattering) for $t > 0$ to the KdV equation

(*) $$q_t + q_{xxx} - 6qq_x = 0$$

with scattering data $\mathcal{S}(t)$ as in (4.2) where the initial values, $\mathcal{S}(0)$, are the scattering data for $Q(x)$.

The proof of the theorem consists of three steps, the first of which (our Lemma 4.2) appears in [7] (Theorem 4.7 and Remark 4.5 thereafter).

<u>Lemma 4.2.</u> If $Q(x) \in L^1_\alpha$, then the function $H(x)$ defined by

(4.3) $$H(x) \equiv C_a M \int_{-a}^{a} \frac{2ik}{\pi} R_+(k,0)e^{2ikx}dk$$

is also in L^1_α, where $R_+(k,0)$ is the reflection coefficient for $Q(x)$ and C_aM denotes the Cesáro mean.

For the sake of completeness, we sketch the proof appearing in [7].

Use the following version of the trace formula:

$$(4.4)\qquad \begin{aligned} Q(x) &= H(x) + 2\int_0^\infty H(x+y)B(x,y)dy + \int_0^\infty H(x+y)(B*B)(x,y)dy \\ &\quad - \sum_{j=1}^{n} 4\, c_j \beta_j e^{-2\beta_j x} m_+^2(x,i\beta_j) \end{aligned}$$

where $B(x,y)$ (resp. $B*B$) is the inverse Fourier transform (in k) of $m_+(x,k) - 1$ (resp. $(m_+(x,k)-1)^2$), and β_j, c_j are the j^{th} bound state and norming constant for $Q(x)$. Recall that, as in Section 2, $Q(x)$ determines $m_+(x,k)$.

Solving (4.4) by iteration for $H(x)$ leads directly to an estimate of the form

$$(4.5)\qquad \|H(\cdot)\|_{L^1_\alpha} \leq C(\alpha)\|Q(\cdot)\|_{L^1_\alpha}$$

with bounded constants $C(\alpha)$ for all $\alpha \geq 1$. Thus $Q(\cdot) \in L^1_\alpha$ implies $H(\cdot) \in L^1_\alpha$.

Remark. The trace formula (4.4) and its time-dependent generalization will play an important role in the discussion of Section 5, in which further properties of the solution constructed in Theorem 4.1 are deduced.

The second step in the proof of Theorem 4.1 concerns the behavior of the Fourier transform of $2ikR_+(k,0) \cdot e^{8ik^3t}$ for $t > 0$ and is crucial to the analysis of Section 5 below.

Lemma 4.3. The function $h(x,t)$ defined by

$$(4.5)\qquad h(x,t) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} 2ikR_+(k,0)e^{2ikx+8ik^3t}dk$$

is well-defined for $t > 0$ and satisfies:

$$
(4.6)\quad \left\{
\begin{array}{ll}
\text{(i)} & h(\cdot,t) \in X^1_{13/4-\delta} \quad \text{for every} \quad t > 0 \quad \text{fixed and every } \delta > 0 \\
 & \text{with} \quad t \to h(\cdot,t) \text{ continuous in this topology} \\
\text{(ii)} & t \to h(\cdot,t) \in C(]0,\infty[,X^\infty_4) \\
 & \text{In fact, } \lim_{x\to+\infty} (1+|x|)^4 h(x,t) = 0 \quad \text{for} \quad t>0 \quad \text{fixed} \\
\text{(iii)} & h(\cdot,t) \in L^\infty_{1/4} \\
\text{(iv)} & \text{All of the above hold for } \partial^r_t\partial^s_x\, h(x,t) \text{ provided we} \\
 & \text{subtract } \tfrac{1}{2}(3r+s) \text{ from every lower index (the poly-} \\
 & \text{nomial weighting) i.e.} \\
 & \partial^r_t\partial^s_x h(x,t) \in X^1_{13/4-\delta-\frac{3r+s}{2}} \quad \text{for every} \quad t>0 \quad \text{fixed} \\
 & \text{etc.} \\
\text{(v)} & h(x,t) \to H(x) \quad \text{in} \quad X^1_{3-\delta} \quad \text{as} \quad t \downarrow 0 \\
\text{(vi)} & h(x,t) = O(t^{-1/3}) \quad \text{as} \quad t \to +\infty \quad \text{for fixed} \quad x. \\
\text{(vii)} & h(ct+\xi,t) = \begin{cases} o(t^{-13/3}) & \text{if} \quad c>0 \\ O(t^{-1/2}) & \text{if} \quad c<0 \end{cases} \\
 & \qquad\qquad \text{as} \quad t \to +\infty \ (\xi \quad \text{fixed}).
\end{array}
\right.
$$

Remark. (iv) implies that for $3r+s \leq 5$,

$$\partial^r_t\partial^s_x\, h(x,t) \quad \text{is continuous.}$$

Thus $h(x,t)$ is, for $t > 0$, a classical solution of the linear equation:

$$h_t + h_{xxx} = 0 \tag{4.7}$$

and (v) describes the sense in which the initial data is taken on.

Proof of Lemma 4.3. We first note that for potentials in L_4^1, the associated reflection coefficient $R_+(k,0)$ is C^3 (see [7]). Given $t > 0$ and x, both fixed and finite, we may pick a finite k_0 with $k_0 > \left(\frac{|x|}{12t}\right)^{1/2}$, which implies $x + 12k^2t > 0$ for $|k| > k_0$. Integrate (4.5) by parts in $|k| > k_0$, bringing down a factor of $x + 12k^2t$ in the denominator, to show that the integral in (4.5) is well-defined for any $t > 0$ and x fixed.

In order to verify the properties of $h(x,t)$, we re-express (4.5), using (4.3) above to replace $2ikR_+(k,0)$ by the inverse Fourier transform of $H(x)$. We then have:

$$(4.8) \qquad h(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(y) e^{2ik(x-y)+8ik^3t} dydk$$

where the y-integral is absolutely convergent and, as remarked above, the iterated integral converges. Thus $h(x,t)$ is the convolution of $H(x)$ with the fundamental solution

$$(4.9) \qquad E(x-y,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{2ik(x-y)+8ik^3t} dk$$

$$= (3t)^{-1/3} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{2iK\left(\frac{x-y}{(3t)^{1/3}}\right) + \frac{8iK^3}{3}} dK$$

$$= (3t)^{-1/3} \mathrm{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right)$$

where $\mathrm{Ai}(z)$ is the well-known Airy function.

We shall make use of the following properties of $Ai(z)$ (c.f. [20] for example):

(a) $Ai''(z) = z\,Ai(z)$

(b) $|Ai(z)| < 1$ for all real z

(c) For $z > 0$, $Ai(z)$ is a non-negative, monotone decreasing function of z with asymptotic rate of decay proportional to $z^{-1/4} \cdot e^{-\zeta}$, while Ai' decays like $z^{+1/4}\, e^{-\zeta}$, where $\zeta \equiv 2/3\, z^{3/2}$

(d) As $z \to -\infty$, $Ai(z)$ is highly oscillatory with an algebraically decaying envelope. In fact,

$$Ai(z) \sim \frac{1}{\pi^{1/2}} |z|^{1/4}\{\cos(\tilde{\zeta} - \pi/4) + O(\tfrac{1}{\tilde{\zeta}})\}$$

where $\tilde{\zeta} = 2/3\, |z|^{3/2}$,

and $$Ai'(z) \sim \frac{|z|^{1/4}}{\pi^{1/2}}\{\sin(\tilde{\zeta} - \pi/4) + O(\tfrac{1}{\tilde{\zeta}})\}.$$

Using (a), corresponding estimates on higher derivatives of Ai hold.

We now show that (4.6) holds:

$$\int_a^\infty (1 + |x|)^{13/4-\delta} |h(x,t)|dx$$

(4.11) $$= \int_a^\infty (1 + |x|)^{13/4-\delta} \left|\frac{1}{\varepsilon}\int_{-\infty}^\infty Ai\left(\frac{x-y}{\varepsilon}\right)H(y)dy\right|dx$$

where $\varepsilon \equiv (3t)^{1/3} > 0$, fixed

$$\leq \int_a^\infty (1 + |x|)^{13/4-\delta}\left\{\left|\frac{1}{\varepsilon}\int_{-\infty}^x Ai\left(\frac{x-y}{\varepsilon}\right)H(y)dy\right| + \left|\frac{1}{\varepsilon}\int_x^\infty Ai\left(\frac{x-y}{\varepsilon}\right)H(y)dy\right|\right\}dx$$

$$\equiv \int_a^\infty (1 + |x|)^{13/4-\delta}\{|h_1(x,t)| + |h_2(x,t)|\}dx.$$

Now we have

$$(4.12) \qquad \varepsilon \cdot \int_a^\infty (1+|x|)^\alpha |h_1(x,t)|dx$$

$$\leq \int_a^\infty (1+|x|)^\alpha \int_{-\infty}^a |Ai(\tfrac{x-y}{\varepsilon})H(y)|dydx + \int_a^\infty (1+|x|)^\alpha \int_a^x |Ai(\tfrac{x-y}{\varepsilon})H(y)|dydx$$

$$= \int_{-\infty}^a dy \int_a^\infty dx |Ai(\tfrac{x-y}{\varepsilon})||H(y)|(1+|x|)^\alpha + \int_a^\infty dy \int_y^\infty dx |Ai(\tfrac{x-y}{\varepsilon})||H(y)|(1+|x|)^\alpha$$

$$\leq \int_{-\infty}^a |H(y)|dy \cdot \int_a^\infty (1+|x|)^\alpha Ai(\tfrac{x-a}{\varepsilon})dx + \int_a^\infty (1+|y|)^\alpha |H(y)|dt \int_0^\infty (1+|z|)^\alpha Ai(z/\varepsilon)dz$$

$$< \infty.$$

We remark that these estimates give a bound on $\|h_1(\cdot,t)\|_{L^1_\alpha(a)}$ for all $\alpha \leq 4$ uniformly in $0 \leq t \leq T < \infty$ by rescaling.

Similarly,

$$(4.13) \qquad \int_a^\infty (1+|x|)^\alpha |h_2(x,t)|dx = \int_a^\infty (1+|x|)^\alpha \frac{1}{\varepsilon} \int_x^\infty |Ai(\tfrac{x-y}{\varepsilon})||H(y)|dydx$$

$$= \int_a^\infty \int_{-\infty}^0 (1+|x|)^\alpha |Ai(z)||H(x-\varepsilon z)|dzdx$$

$$\leq \int_a^\infty \int_{-\infty}^0 (1+|x-\varepsilon z|)^\alpha |Ai(z)||H(x-\varepsilon z)|dzdx \qquad \text{(assuming w.}\ell\text{.o.g. } a > 0)$$

$$\leq \int_{-\infty}^0 |Ai(z)| \int_{a-\varepsilon z}^\infty (1+|\zeta|)^\alpha |H(\zeta)|d\zeta dz$$

$$\leq \int_{-\infty}^0 |Ai(z)| \cdot (1+|a-\varepsilon z|)^{\alpha-4} dz \cdot \|H(\cdot)\|_{L^1_4(a)}$$

$< \infty$ by the decay rate on Ai if $\alpha < 13/4$.

Note that this is <u>not</u> uniform as $\varepsilon \downarrow 0$.

To estimate $(1+|x|)^4 h(x,t)$, we proceed as follows $(0 < \hat{\delta} < 1)$:

$$(4.14) \qquad (1+|x|)^4 h(x,t) = (1+|x|)^4 \int_{-\infty}^{x(1-\hat{\delta})} \frac{1}{\varepsilon} \mathrm{Ai}\Big(\frac{x-y}{\varepsilon}\Big) H(y)dy$$

$$+ (1+|x|)^4 \int_{x(1-\hat{\delta})}^{\infty} \frac{1}{\varepsilon} \mathrm{Ai}\Big(\frac{x-y}{\varepsilon}\Big) H(y)dy.$$

The second term goes to 0 as $x \to +\infty$, since $(1+|y|)^4 H(y) \in L^1$ and $\mathrm{Ai} \in L^\infty$. In the first term, $y \le (1-\hat{\delta})x$ so $\frac{x-y}{\varepsilon} \ge \frac{\hat{\delta}x}{\varepsilon}$. By monotonicity, $\mathrm{Ai}\big(\frac{x-y}{\varepsilon}\big) \le \mathrm{Ai}\big(\frac{\hat{\delta}x}{\varepsilon}\big)$, which decays exponentially as $x \to +\infty$. Thus $\lim_{x \to +\infty} (1+|x|)^4 h(x,t) = 0$ for every $t > 0$.

To obtain decay of $h(x,t)$ as $x \to -\infty$, we make a similar estimate:

$$(4.15) \qquad (1+|x|)^{1/4} |h(x,t)| = (1+|x|)^{1/4} \Big| \int_{-\infty}^{x(1-\hat{\delta})} \frac{1}{\varepsilon} \mathrm{Ai}\Big(\frac{x-y}{\varepsilon}\Big) H(y)dy \Big|$$

$$+ (1+|x|)^{1/4} \Big| \int_{x(1-\hat{\delta})}^{\infty} \mathrm{Ai}\Big(\frac{x-y}{\varepsilon}\Big) H(y)dy.$$

As $x \to -\infty$, the first term is $O((1+|x|)^{-13/4})$ since $H(\cdot) \in L^1_4$ while for the latter term $(1+|x|)^{1/4} |\mathrm{Ai}\big(\frac{x-y}{\varepsilon}\big)|$ is bounded and $H(\cdot) \in L^1$. Hence $(1+|x|)^{1/4} |h(x,t)| \in L^\infty$ for every $t > 0$.

The extension of these estimates to x and t derivatives, as well as their continuity in t for $t > 0$, follows from the analyticity and estimates on the Airy function.

Asymptotics in t are derived as follows:

$$(4.16) \qquad (3t)^{1/3} h(x,t) = \int_{-\infty}^{\infty} \mathrm{Ai}\Big(\frac{x-y}{(3t)^{1/3}}\Big) H(y)dy.$$

Hence, as $t \to +\infty$ with x fixed (or even varying in a compact set), $(3t)^{1/3}h(x,t)$ is bounded, thus $h(x,t) = O(t^{-1/3})$ as $t \to +\infty$. Now consider $h(ct+\xi,t)$ for $c < 0$:

$$(4.17) \qquad \left|(3t)^{1/3}h(ct+\xi,t)\right| \leq \left|\int_{-\infty}^{\frac{ct}{2}+\xi} \mathrm{Ai}\left(\frac{ct+\xi-y}{(3t)^{1/3}}\right)H(y)dy\right|$$

$$+ \left|\int_{\frac{ct+\xi}{2}}^{\infty} \mathrm{Ai}\left(\frac{ct+\xi-y}{(3t)^{1/3}}\right)H(y)dy\right|$$

$$\leq \int_{-\infty}^{\frac{ct}{2}+\xi} |H(y)|dy + \int_{\frac{ct}{2}+\xi}^{\infty} \left|\mathrm{Ai}\left(\frac{ct+\xi-y}{(3t)^{1/3}}\right)\right| \cdot |H(y)|dy$$

$$\leq \left(1+\left|\frac{ct}{2}+\xi\right|\right)^{-4} \int_{-\infty}^{\frac{ct}{2}+\xi} (1+|y|)^{4} \, |1+(y)|dy$$

$$+ \sup_{z<\frac{ct}{2}} \left|\mathrm{Ai}\left(\frac{z}{(3t)^{1/3}}\right)\right| \cdot \int_{\frac{ct}{2}+\xi}^{\infty} |H(y)|dy$$

$= O(t^{-1/6})$ as $t \to +\infty$ since $\mathrm{Ai}(x) \sim |x|^{-1/4}$ as $x \to -\infty$.

i.e. $h(ct+\xi,t) = O(t^{-1/2})$ for $c < 0$.

Note that this agrees with the alternate method of estimate, namely stationary phase analysis of (4.5).

For $c > 0$, we have:

$$(4.18) \qquad |(3t)^{1/3}h(ct+\xi)| \leq \int_{-\infty}^{\frac{ct}{2}+\xi} |Ai(\frac{ct+\xi-y}{(3t)^{1/3}})||1+(y)|dy$$

$$+\int_{\frac{ct}{2}+\xi}^{\infty} |Ai(\frac{ct+\xi-y}{(3t)^{1/3}})||H(y)|dy$$

$$\leq \sup_{z\geq\frac{ct}{2}} |Ai(\frac{z}{(3t)^{1/3}})| \int_{-\infty}^{\frac{ct}{2}+\xi} |H(y)|dy$$

$$+\frac{1}{(1+|\frac{ct}{2}+\xi|)^4} \int_{\frac{ct}{2}+\xi}^{\infty} (1+|y|)^4|H(y)|dy = o(t^{-4}),$$

hence we have in this case $h(ct+\xi,t) = o(t^{-13/3})$ as $t \to +\infty$.

To show that $h(x,t) \to H(x)$ in $X'_{3-\delta}$ as $t \downarrow 0$, first we consider (as usual $\varepsilon \equiv (3t)^{1/3}$)

$$(4.19) \qquad \int_{-\frac{1}{\varepsilon}^{1/2}}^{\frac{1}{\varepsilon}^{1/2}} Ai(y)[H(x-\varepsilon y) - H(x)]dy.$$

Now $H(\cdot) \in L_4^1$ implies that, since translation is continuous,

$\|H(x-\varepsilon y) - H(x)\|_{L_4^1} = o(\|\varepsilon y\|)$ as $|\varepsilon y| \to 0$.

On the given interval, $|\varepsilon y| \leq \varepsilon^{1/2}$ while the interval length is $2\varepsilon^{-1/2}$.
Thus the expression (4.19) is $o(1)$ as $\varepsilon \to 0$ as an element in L_4^1.

Now we have:

$$(4.20) \qquad h(x,t) = \int_{-\infty}^{-\frac{1}{\varepsilon}^{1/2}} Ai(y)\ H(x-\varepsilon y)dy$$

$$+\int_{-\frac{1}{\varepsilon}^{1/2}}^{\frac{1}{\varepsilon}^{1/2}} Ai(y)H(x-\varepsilon y)dy + \int_{\frac{1}{\varepsilon}^{1/2}}^{\infty} Ai(y)H(x-\varepsilon y)dy.$$

The first term is estimated in $L^1_{3-\delta}(a)$ by:

$$(4.21)\qquad \int_a^\infty \int_{-\infty}^{-\frac{1}{\varepsilon}^{1/2}} |Ai(y)|(1+|x|)^{3-\delta}|H(x-\varepsilon y)|dydx$$

$$\leq C|\varepsilon|^{1/8} \cdot \int_a^\infty \int_{-\infty}^{-\frac{1}{\varepsilon}^{1/2}} (1+|x|)^{3-\delta}|H(x-\varepsilon y)|dydx$$

$$\leq C|\varepsilon|^{1/8} \int_a^\infty (1+|x|)^{3-\delta} \int_{x+\varepsilon^{1/2}}^\infty |H(z)|dzdx$$

$$\leq C|\varepsilon|^{1/8} \int_a^\infty (1+|x|)^{-1-\delta}dx \cdot \|H(\cdot)\|_{L^1_4(a)}$$

$$= O(|\varepsilon|^{1/8}).$$

Now

$$\int_a^\infty \int_{\frac{1}{\varepsilon}^{1/2}}^\infty |Ai(y)|(1+|x|)^{3-\delta}|H(x-\varepsilon y)|dydx$$

$$\leq \int_a^\infty \int_{\frac{1}{\varepsilon}^{1/2}}^\infty |Ai(y)|(1+|x-\varepsilon y|)^{3-\delta}(1+|\varepsilon y|)^{3-\delta}|H(x-\varepsilon y)|dydx$$

$$\leq \|H(\cdot)\|_{L^1_{3-\delta}} \int_{\frac{1}{\varepsilon}^{1/2}}^\infty |Ai(y)|(1+|\varepsilon y|)^{3-\delta}dy$$

which tends to 0 exponentially fast as $\varepsilon \downarrow 0$.

Thus

$$h(x,t) \to \lim_{\varepsilon\downarrow 0} \int_{-\frac{1}{\varepsilon}^{1/2}}^{\frac{1}{\varepsilon}^{1/2}} Ai(y)H(x)dy = H(x)$$

in $X'_{3-\delta}$ as $t \downarrow 0$.

This completes the proof of Lemma 4.3.

<u>Remark.</u> If we assume more generally $Q(x) \in L^1_\sigma$, then $H(x) \in L^1_\sigma$ and the corresponding results for $h(x,t)$ are:

(4.21) (i) $h(\cdot,t)\ X'_{\sigma-3/4-\delta} \cap X^\infty_\sigma \cap L^\infty_{1/4}$

for $t > 0$ fixed, with $t \to h(\cdot,t)$ continuous for $t > 0$.

(ii) $$h(ct+\xi,t) = \begin{cases} O(t^{-1/2}) & \text{for } c < 0 \\ o(t^{-\sigma-1/3}) & \text{for } c > 0 \end{cases}$$

(iii) $h(\cdot,t) \to H(\cdot)$ in $X'_{\sigma-1-\delta}$ as $t \to +\infty$ (ξ fixed).

In particular, if $Q \in L^1$ with compact support, then $h(x,t)$ is C^∞ for $t > 0$, h decays faster than any power of x as $x \to +\infty$ for fixed t and h decays faster than any power of t in moving frames with $c > 0$ as $t \to +\infty$ (c.f. [6]). Better decay as $x \to -\infty$ and in moving frames for $c < 0$ is related to additional smoothness of $Q(x)$ and hence $H(x)$.

Using Lemma 4.3 and the results of [7] (c.f. (2.17) above), it follows that for every $t > 0$, a potential $q(x,t)$ may be recovered via inverse scattering starting from $+\infty$ and solving to the left for every finite x. Thus a generalized solution (in the sense of inverse scattering) exists and, for each fixed $t > 0$, $q(\cdot,t) \in X'_{13/4-\delta}$. In the next section, we strengthen this result to include smoothness and decay of the generalized solution.

5. Regularity and Decay of the Generalized Solution.

By analyzing the inverse scattering method of Deift and Trubowitz [7] more closely, smoothness and spatial decay properties of the generalized solution constructed above will be obtained. The solution behaves quite similarly to $h(x,t)$, as discussed in Lemma 4.3 above, except for the presence of a finite sum of "soliton" terms. We shall prove the following theorem.

Theorem 5.1. The generalized solution $q(x,t)$ constructed above satisfies:

$$(5.1)\quad \begin{cases} \text{(i)} & q(\cdot,t) \in X^1_{13/4-\delta} \cap X^\infty_4 \quad \text{for } t > 0 \\ & \text{and } t \to q(\cdot,t) \text{ is continuous for } t > 0 \\ \text{(ii)} & \partial_t^r \partial_x^s q(\cdot,t) \in X^1_{13/4-\delta-\left(\frac{s+3r}{2}\right)} \cap X^\infty_{4-\left(\frac{3r+s}{2}\right)} \\ & \text{(with continuity in } t) \\ \text{(iii)} & q(x,t) \to Q(x) \quad \text{in } X^1_{3-\delta}. \end{cases}$$

Our proof is essentially a parameterized (by t) version of the Deift-Trubowitz approach to inverse scattering [7], sketched in Section 2 above. We begin by rewriting the time-dependent analogues of the trace formula (2.14) and the Volterra equation for m_+ (2.2) in terms of $b(x,y,t)$, which is the inverse Fourier transform of $m_-(x,k,t)-1$ with respect to k. Since m_+-1 belongs to the Hardy space H^{2^+}, $b(x,y,t) = 0$ for $y < 0$. We have the Volterra equation

$$(5.2)\quad b(x,y,t) = \int_{x+y}^{\infty} q(z,t)dz + \int_0^y dz \int_{x+y-z}^{\infty} dw\, q(w,t)b(w,z,t), \quad y > 0$$

and the "trace formula"

$$(5.3)\qquad q(x,t) = \tilde{h}(x,t) + 2\int_0^\infty \tilde{h}(x+y,t)b(x,y,t)dy$$

$$+ \int_0^\infty \tilde{h}(x+y,t)(b*b)(x,y,t)dy$$

$$\tilde{h}(x,t) \equiv h(x,t) - \sum_{j=1}^{n} 4\beta_j c_j(0)e^{-2\beta_j x+8\beta_j^3 t}$$

where, as in Section 4 above, $h(x,t)$ is the Fourier transform of $2ikR_+(k,0)e^{8ik^3t}$ (see (4.5) above). Note that the addition of the exponential terms does not alter the results of Lemmas 4.2, 4.3. We regard (5.2), (5.3) as the fundamental equations to be solved, and as in [7], we solve them by iteration to the right of a sufficiently large point (depending on $\tilde{h}(x,t)$) and then argue that the solution continues to exist for all finite x values, by using a local Lipschitz estimate.

One important step in this procedure is to obtain estimates on the solution of (5.2) if we assume $q(x,t)$ is known. We introduce auxiliary functions $\eta(x,t)$, $\gamma(x,t)$ assuming $q(\cdot,t) \in X_1^1$ for every $t > 0$, as follows:

$$(5.4)\qquad \begin{cases} \eta(x,t) \equiv \int_x^\infty |q(y,t)|dy \\ \gamma(x,t) \equiv \int_x^\infty (y-x)|q(y,t)|dy. \end{cases}$$

Note that η,γ are non-increasing functions of x for each fixed t which are finite for all $x > -\infty$ and that

$$(5.5)\qquad \begin{aligned} \int_x^\infty \eta(y,t)dy &= \int_x^\infty dy \int_y^\infty dz\ |q(z,t)| \\ &= \int_x^\infty dz|q(z,t)|(z-x) \equiv \gamma(x,t). \end{aligned}$$

Similarly, if $\frac{\partial}{\partial t} q(\cdot,t) \in X_1^1$, we may introduce

$$(5.6)\qquad \begin{cases} \lambda(x,t) \equiv \int_x^\infty |\frac{\partial q}{\partial t}(y,t)|dy \\ \\ \mu(x,t) \equiv \int_x^\infty (y-x)|\frac{\partial q}{\partial t}(y,t)|dy. \end{cases}$$

With these functions thus defined, we prove the analogue of Lemma 2.3 of [7].

Lemma 5.2. If $q(\cdot,t) \in X_1^1$ for every $t \geq 0$ and $\frac{\partial q}{\partial t}(\cdot,t) \in X_1^1$ for every $t \geq 0$, the integral equation (5.2) has a unique solution $b(x,y,t)$ satisfying:

(5.7)

(i) $|b(x,y,t)| \leq e^{\gamma(x,t)}\eta(x+y,t)$

(ii) $b(x,y,t)$ is absolutely continuous in x and y with

$$|\frac{\partial b}{\partial y}(x,y,t) + q(x+y,t)| \leq e^{\gamma(x,t)}\eta(x,t) \cdot \eta(x+y,t)$$

$$|\frac{\partial b}{\partial y}(x,y,t) + q(x+y,t)| \leq 2e^{\gamma(x,t)}\eta(x,t)\eta(x + y,t)$$

(iii) b solves the 'wave equation'

$$\frac{\partial^2}{\partial x^2}b(x,y,t) - \frac{\partial^2}{\partial x\partial y} b(x,y,t) = q(x,t)b(x,y,t)$$

with $-\frac{\partial}{\partial x} b(x,0^+,t) = -\frac{\partial}{\partial y} b(x,0^+,t) = q(x,t)$

(iv) $|\frac{\partial b}{\partial t}(x,y,t)| \leq e^{\gamma(x,t)}\{\lambda(x+y,t) + e^{\gamma(x,t)} \cdot \eta(x+y,t)\mu(x,t)\}$

(v) If $(\frac{\partial}{\partial x})^j q(x,t) \in X_1^1$ for every $t > 0$ and $0 \leq j \leq \ell$ then b has mixed deriatives in x and y up to order $\ell+1$ obeying estimates analogous to (ii) above.

Proof. (i), (ii) and (iii) are proved in Lemma 2.3 of [7], so we merely sketch their proof.

Solving (5.2) by iteration with

$$(5.8)\qquad \begin{cases} b(x,y,t) = \sum_{j=0}^{\infty} b_j(x,y,t) \\ \\ b_0(x,y,t) = \int_{x+y}^\infty q(z,t)dz \\ \\ b_{j+1}(x,y,t) = \int_0^y dz \int_{x+y-z}^\infty dw\, q(w,t)b_j(w,z,t) \end{cases}$$

leads to the inductive estimate [7]

$$|b_j(x,y,t)| \leq \frac{(\gamma(x,t))^j}{j!} \eta(x+y,t)$$

which proves (i).

Differentiating (5.2) and using the above estimate, we have

$$(5.10)\quad \left\{\begin{aligned} &\left|\frac{\partial}{\partial x} b(x,y,t) + q(x+y,t)\right| \\ &= \left| \int_0^y dz\, q(x+y-z,t)b(x+y-z,z,t)\right| \\ &\leq \int_0^y dz\,|q(x+y-z,t)|e^{\gamma(x+y-z,t)}\eta(x+y,t) \\ &\leq e^{\gamma(x,t)}\eta(x+y,t) \int_0^y |q(x+y-z,t)| \\ &\leq e^{\gamma(x,t)}\eta(x+y,t)\eta(x,t) \end{aligned}\right.$$

with a similar estimate for the y derivative.

(iii) is a direct calculation.

To establish (iv), differentiate (5.2) with respect to t, to obtain

$$(5.11)\quad \frac{\partial b}{\partial t}(x,y,t) = \int_{x+y}^{\infty} \frac{\partial q}{\partial t}(z,t)dz$$

$$+ \int_0^{\infty} dz \int_{x+y-z}^{\infty} dw\, \left(\frac{\partial b}{\partial t}(w,z,t)q(w,t) + b(w,z,t)\frac{\partial q}{\partial t}(w,t)\right).$$

Again we solve by iteration, writing

$$(5.12)\quad \left\{\begin{aligned} &b_t(x,y,t) = \sum_{j=0}^{\infty} \zeta_j(x,y,t) \\ &\zeta_0(x,y,t) = \int_{x+y}^{\infty} dz\, \frac{\partial q}{\partial t}(z,t) \\ &\qquad + \int_0^y dz \int_{x+y-z}^{\infty} dw\, b(w,z,t)\, \frac{\partial q}{\partial t}(w,t) \\ &\zeta_{j+1}(x,y,t) = \int_0^y dz \int_{x+y-z}^{\infty} dw\, q(w,t)\zeta_j(w,z,t). \end{aligned}\right.$$

Then $$|\zeta_0(x,y,t)| \leq \lambda(x+y,t) + \int_0^y dz \int_{x+y-z}^{\infty} dw \left|\frac{\partial q}{\partial t}(w,t)\right| e^{\gamma(w,t)} \eta(w+z,t)$$

$$\leq \lambda(x+y,t) + \eta(x+y,t)\mu(x,t) \cdot e^{\gamma(x,t)}$$

and

$$|\zeta_n(x,y,t)| \leq |\zeta_0(x,y,t)| \cdot \frac{(\gamma(x,t))^n}{n!}$$

(established just as before), which proves (iv). (v) is proved inductively by applying the same ideas as in (ii) to higher derivatives. Namely, from the relation

$$(5.13) \qquad \left(\frac{\partial}{\partial x}\right)^{j+1} b(x,y,t) + \left(\frac{\partial}{\partial x}\right)^{j} q(x+y,t)$$

$$= -\int_0^y \left(\frac{\partial}{\partial x}\right)^{j} (q(x+y-z,t)b(x+y-z,z,t))dz$$

and estimates on derivatives of q and b of lower order, we can readily estimate $\left(\frac{\partial}{\partial x}\right)^{j+1} b(x,y,t)$. This completes the proof of Lemma 5.2.

We shall also require estimates on the difference between the functions $b^{(j)}(x,y,t)$, $j = 1,2$, corresponding to two potentials $q^{(1)},q^{(2)}$. Define

$$\delta\eta(x,t) \equiv \int_x^{\infty} |q^{(1)}(y,t) - q^{(2)}(y,z)|dy$$

$$(5.14) \qquad \delta\gamma(x,t) = \int_x^{\infty} (y-x)|q^{(1)}(y,t) - q^{(2)}(y,t)|dy$$

$$\delta\lambda(x,t) \equiv \int_x^{\infty} \left|\frac{\partial q^{(1)}}{\partial t}(y,t) - \frac{\partial q^{(2)}}{\partial t}(y,t)\right|dy$$

$$\delta\mu(x,t) \equiv \int_x^{\infty} (y-x)\left|\frac{\partial q^{(1)}}{\partial t}(y,t) - \frac{\partial q^{(2)}}{\partial t}(y,t)\right|dy$$

for notational ease. We have:

<u>Lemma 5.3.</u> (c.f. Lemma 2.4 of [7]) If $q^{(1)}(x,t)$, $q^{(2)}(x,t)$ are two potentials as in Lemma 5.2 above, then the corresponding solutions $b^{(1)}(x,y,t)$, $b^{(2)}(x,y,t)$ of (5.2) satisfy:

$$
(5.15)\quad\begin{cases}
\text{(i)} & |b^{(1)}(x,y,t) - b^{(2)}(x,y,t)| \le e^{\gamma_1(x,t)}(1+\gamma_2(x,t)e^{\gamma_2(x,t)}) \\
& \qquad \cdot\ \delta\eta(x+y,t) \\
\text{(ii)} & |b^{(1)}(x,y,t) - b^{(2)}(x,y,t)| \le e^{\gamma_1(x,t)} \cdot \delta\eta(x+y,t) \\
& + e^{\gamma_1(x,t)+\gamma_2(x,t)}\eta_2(x+y,t) \cdot \delta\gamma(x,t) \\
\text{(iii)} & |\frac{\partial b^{(1)}}{\partial t}(x,y,t) - \frac{\partial b^{(2)}}{\partial t}(x,y,t)| \le e^{\gamma_1(x,t)}\left(1+\gamma_2(x,t)e^{\gamma_2(x,t)}\right) \\
& (\delta\lambda(x+y,t) + \mu_2(x,t)e^{\gamma_2(x,t)} \cdot \delta\eta(x+y,t)) \\
\text{(iv)} & |\frac{\partial b^{(1)}}{\partial t}(x,y,t) - \frac{\partial b^{(2)}}{\partial t}(x,y,t)| \le \\
& e^{\gamma_1(x,t)}\{\delta\lambda(x+y,t) + e^{\gamma_2(x,t)}[\eta_2(x+y,t)\delta\mu(x,t) \\
& + \delta\gamma(x,t)e^{\gamma_2(x,t)}(\lambda_2(x+y,t) + e^{\gamma_2(x,t)}\eta_2(x+y,t)\mu_2(x,t))]\}.
\end{cases}
$$

Remark. (i) and (iii) provide L^∞ estimates in y while (ii) and (iv) lead to L^1 estimates for $0 \le y \le \infty$.

Proof of Lemma 5.3. Once again we sketch the first two estimates and refer to [7] for details. Subtracting (5.2) for $b^{(2)}$ from the equation for $b^{(1)}$ we have

$$(5.16)\qquad \rho(x,y,t) \equiv b^{(1)}(x,y,t) - b^{(2)}(x,y,t)$$

$$= \int_{x+y}^{\infty} (q^{(1)}(z,t) - q^{(2)}(z,t))dz$$

$$+ \int_0^y dz \int_{x+y-z}^{\infty} dw\{q^{(1)}(w,t)(b^{(1)}(w,z,t) - b^{(2)}(w,z,t))$$

$$+ (q^{(1)}(w,t) - q^{(2)}(w,t))b^{(2)}(w,z,t)\}.$$

Solving for $\rho = \sum_{j=0}^{\infty} \rho_j(x,y,t)$ leads to the usual inductive estimate

$$(5.17)\qquad |\rho_j(x,y,t)| \le |\rho_0(x,y,t)| \cdot \frac{(\gamma_1(x,t))^j}{j!}$$

where

$$(5.18)\qquad \rho_0(x,y,t) = \int_{x+y}^{\infty} (q^{(1)}(z,t) - q^{(2)}(z,t))dz$$

$$+ \int_0^y dz \int_{x+y-z}^{\infty} dw(q^{(1)}(w,t) - q^{(2)}(w,t))b^{(2)}(w,z,t).$$

Thus

$$|\rho_0(x,y,t)| \le \int_{x+y}^{\infty} |q^{(1)}(z,t) - q^{(2)}(z,t)|dz$$

$$+ \int_0^y dz \int_{x+y-z}^{\infty} dw|q^{(1)}(w,t) - q^{(2)}(w,t)|e^{\gamma_2(w,t)}\eta_2(w+z,t).$$

(i) follows by interchanging the order in the double integral and integrating $\eta_2(w+z,t)$ while (ii) comes from using the monotonicity of η_2 which yields the $\eta_2(x+y,t)$ factor.

(iii) and (iv) are derived similarly. Differentiating (5.16) with respect to t gives

$$(5.19)\quad \frac{\partial b^{(1)}}{\partial t}(x,y,t) - \frac{\partial b^{(2)}}{\partial t}(x,y,t) = \int_{x+y}^{\infty} \left(\frac{\partial q^{(1)}}{\partial t}(z,t) - \frac{\partial q^{(2)}}{\partial t}(z,t)\right)$$

$$+ \int_0^y dz \int_{x+y-z}^{\infty} dw \left\{ \begin{array}{l} \frac{\partial q^{(1)}}{\partial t}(w,t)[b^{(1)}(w,z,t) - b^{(2)}(w,z,t)] \\ + q^{(1)}(w,t)\ [\frac{\partial b^{(1)}}{\partial t}(w,z,t) - \frac{\partial b^{(2)}}{\partial t}(w,z,t)] \end{array}\right.$$

$$+ \left(\frac{\partial q^{(1)}}{\partial t}(w,t) - \frac{\partial q^{(2)}}{\partial t}(w,t)\right) b^{(2)}(w,z,t)$$

$$+ (q^{(1)}(w,t) - q^{(2)}(w,t))\ \frac{\partial b^{(2)}}{\partial t}(w,z,t)\Big\}.$$

Solving iteratively for $\frac{\partial b^{(1)}}{\partial t} - \frac{\partial b^{(2)}}{\partial t}$ again leads to an inductive estimate involving $(\gamma_1(x,t))^j/j!$, but now the leading term is

$$\int_{x+y}^{\infty} \left(\frac{\partial q^{(1)}}{\partial t}(z,t) - \frac{\partial q^{(2)}}{\partial t}(z,t)\right)dz$$

$$+ \int_0^y dz \int_{x+y-z}^{\infty} dw \left\{ \begin{array}{l} \frac{\partial q^{(1)}}{\partial t}(w,t)[b^{(1)}(w,z,t) - b^{(2)}(w,z,t)] \\ + \left(\frac{\partial q^{(1)}}{\partial t}(w,t) - \frac{\partial q^{(2)}}{\partial t}(w,t)\right) b^{(2)}(w,z,t) \end{array}\right.$$

$$+ (q^{(1)}(w,t) - q^{(2)}(w,t))\frac{\partial b^{(2)}}{\partial t}(w,z,t) \quad .$$

Using (i), (ii) and the estimates of Lemma 5.2, depending on whether we integrate out the bounds for the b's or take their suprema over z, we obtain (iii) and (iv) above. This proves the lemma.

The above lemmas will be used to show, in a manner analogous to [7], that the mapping

$$q \mapsto \Phi(q) \tag{5.20}$$

given by finding b from q via (5.2) and then defining $\Phi(q)$ from b via (5.3) is a contraction on a suitable ball in a Banach space. In [7], this space is $\{Q(x) : \int_a^\infty |x||Q(x)|dx \leq 1\}$ where a is chosen so that $\int_a^\infty |x||H(x)|dx$ is sufficiently small (recall $H(x)$ is the inverse Fourier transform of $2ikR_+(k,0)$). Since we are interested in smoothness and decay properties of our generalized solution, we shall include these in our Banach space.

Before discussing the mapping $q \mapsto \Phi(q)$ more fully, we require some estimates on the relation (5.3), where we regard $b(x,y,t)$ as known.

Lemma 5.4. Suppose functions $\gamma(x,t)$, $\eta(x,t)$, $\lambda(x,t)$, $\mu(x,t)$ exist satisfying (5.5) and its analogue for λ,μ and such that the estimates (5.7) (i) and (iv) hold. Then if $q(x,t)$ is given by (5.3), we have, for every $M > 0$

$$\text{(5.21)}\quad \left\{\begin{aligned}
&\text{(i)} && \|q(\cdot,t)\|_{L^1_\alpha(M)} \leq (1+\gamma(M,t)e^{\gamma(M,t)})^2 \cdot \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} \\
&\text{(ii)} && \|q(\cdot,t)\|_{L^\infty_\alpha(M)} \leq (1+\gamma(M,t)e^{\gamma(M,t)})^2 \|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)} \\
&\text{(iii)} && \left\|\frac{\partial q}{\partial t}(\cdot,t)\right\|_{L^1_\alpha(M)} \leq (1+\gamma(M,t)e^{\gamma(M,t)})^2 \left\{\left\|\frac{\partial \tilde{h}}{\partial t}(\cdot,t)\right\|_{L^1_\alpha(M)} \right. \\
& && \qquad \left. + 2\mu(M,t)e^{\gamma(M,t)}\|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)}\right\} \\
&\text{(iv)} && \left\|\frac{\partial q}{\partial x}(\cdot,t)\right\|_{L^1_\alpha(M)} \leq \left(1+\gamma(M,t)e^{\gamma(M,t)}\right)^2 \left\|\frac{\partial \tilde{h}}{\partial x}(\cdot,t)\right\|_{L^1_\alpha(M)} \\
& && \qquad + 2\gamma(M,t)(1+\gamma(M,t)e^{\gamma(M,t)})(1+\eta(M,t)e^{\gamma(M,t)})\|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)}.
\end{aligned}\right.$$

Proof of Lemma 5.4. From (5.3),

$$q(x,t) = \tilde{h}(x,t) + 2\int_0^\infty \tilde{h}(x+y,t)b(x,y,t)dy + \int_0^\infty \tilde{h}(x+y,t)(b*b)(x,y,t)dy.$$

Thus

$$(5.22)\qquad \int_M^\infty (1+|x|)^\alpha |q(x,t)|dx \le \int_M^\infty (1+|x|)^\alpha |\tilde{h}(x,t)|dx$$

$$+ \int_M^\infty (1+|x|)^\alpha \left|2\int_0^\infty \tilde{h}(x+y,t)b(x,y,t)dy\right|dx$$

$$+ \int_M^\infty (1+|x|^\alpha)\left|\int_0^\infty \tilde{h}(x+y,t)(b*b)(x,y,t)dy\right|dx$$

$$\le \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} + 2\int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)|e^{\gamma(x,t)}\eta(x+y,t)dydx$$

$$+ \int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)||b*b(x,y,t)|dydx$$

$$\le \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} + 2\int_M^\infty \eta(x,t)e^{\gamma(x,t)} \int_0^\infty (1+|x+y|)^\alpha |\tilde{h}(x+y,t)|dydx$$

$$+ \int_M^\infty \gamma(x,t)\eta(x,t)e^{2\gamma(x,t)} \int_0^\infty (1+|x+y|)^\alpha |\tilde{h}(x+y,t)|dydx$$

$$\le \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} \{1 + 2\gamma(M,t)e^{\gamma(M,t)} + \gamma^2(M,t)e^{2\gamma(M,t)}\}.$$

Similarly,

$$(5.23\qquad \|q(\cdot,t)\|_{L^\infty_\alpha(M)} = \sup_{M\le x\le\infty} (1+|x|)^\alpha |q(x,t)|$$

$$\le \sup_{M\le x\le\infty} (1+|x|)^\alpha |\tilde{h}(x,t)| + 2\sup_{M\le x\le\infty} (1+|x|)^\alpha \cdot \int_0^\infty |\tilde{h}(x+y,t)||b(x,y,t)|dy$$

$$+ \sup_{M\le x\le\infty} (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)||(b*b)|(x,y,t)dy$$

$$\leq \|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)} + 2 \sup_{M\leq x\leq\infty} (1+|x|)^\alpha \sup_{0\leq y\leq\infty} |\tilde{h}(x+y,t)| \cdot \gamma(x,t)e^{\gamma(x,t)}$$

$$+ \sup_{M\leq y\leq\infty} (1 + |x|)^\alpha \cdot \sup_{0\leq y\leq\infty} |\tilde{h}(x+y,t)| \cdot \gamma^2(x,t)e^{2\gamma(x,t)}$$

$$\leq \|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)} (1+\gamma(x,t)e^{\gamma(x,t)})^2.$$

To estimate $\frac{\partial q}{\partial t}$, we note that

$$(5.24) \qquad \frac{\partial q}{\partial t}(x,t) = \frac{\partial \tilde{h}}{\partial t}(x,t) + 2\int_0^\infty \frac{\partial \tilde{h}}{\partial t}(x+y,t)b(x,y,t)dy$$

$$+ \int_0^\infty \frac{\partial \tilde{h}}{\partial t}(x+y,t)(b*b)(x,y,t)dy$$

$$+ 2\int_0^\infty \tilde{h}(x+y,t)\frac{\partial b}{\partial t}(x,y,t)dy + \int_0^\infty \tilde{h}(x+y,t)\frac{\partial}{\partial t}(b*b)(x,y,t)dy.$$

The $\frac{\partial \tilde{h}}{\partial t}$ terms are estimated as above, and the $\frac{\partial b}{\partial t}$ terms are easily seen to provide the remaining terms in the estimate (5.21)(iii) above.

Finally, we consider $\frac{\partial q}{\partial x}$:

$$(5.25) \qquad \frac{\partial q}{\partial x}(x,t) = \frac{\partial \tilde{h}}{\partial x}(x,t) + 2\int_0^\infty \frac{\partial \tilde{h}}{\partial x}(x+y,t)b(x,y,t)dy$$

$$+ \int_0^\infty \frac{\partial \tilde{h}}{\partial x}(x+y,t)(b*b)(x,y,t)dy$$

$$+ 2\int_0^\infty \tilde{h}(x+y,t)\frac{\partial b}{\partial x}(x,y,t)dy$$

$$+ 2\int_0^\infty \tilde{h}(x+y,t)(b * \frac{\partial b}{\partial x})(x,y,t)dy.$$

Again the $\frac{\partial \tilde{h}}{\partial x}(x,t)$ terms lead to estimates with a factor $(1 + \gamma(M,t)e^{\gamma(M,t)})^2$.

The remaining terms (those involving $\frac{\partial b}{\partial x}$) lead to bounds as follows:

(5.26) $$\int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)||\frac{\partial b}{\partial x}(x,y,t)|dydx$$

$$+ \int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)||b * \frac{\partial b}{\partial x}(x,y,t)|dydx$$

$$\leq \int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)|\{|\tilde{q}(x+y,t)|$$

$$+ e^{\gamma(x,t)}\eta(x+y,t)\eta(x,t)\}dydx \cdot (1 + \sup_y O(x,y,t)|)$$

$$\leq \int_M^\infty (1+|x|)^\alpha \sup_{0\leq y\leq\infty} |\tilde{h}(x+y,t)|\{ \int_x^\infty|\tilde{q}(z,t)|dz$$

$$+ e^{\gamma(x,t)}\gamma(x,t)\eta(x,t)\} \cdot (1+\eta(x,t)e^{\gamma(x,t)})dx$$

$$= \int_M^\infty (1+|x|)^\alpha \sup_{0\leq y\leq\infty} |\tilde{h}(x+y,t)| \cdot \eta(x,t)(1+\gamma(x,t)e^{\gamma(x,t)})$$

$$\cdot (1+\eta(x,t)e^{\gamma(x,t)})dx$$

$$\leq \|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha}(1+\gamma(M,t)e^{\gamma(M,t)})(1+\eta(M,t)e^{\gamma(M,t)}) \cdot \gamma(M,t).$$

This completes the proof of Lemma 5.4.

We shall need estimates on the difference between two potentials $q^{(1)}(x), q^{(2)}(x)$ given by (5.3) from $b^{(1)}, b^{(2)}$ respectively.

<u>Lemma 5.5.</u> Let $b^{(1)}, b^{(2)}$ be as in Lemma 5.4 above and satisfy the estimates (5.15) (i)-(iv). Then if $q^{(j)}$ are given by (5.3) with $b = b^{(j)}$, we have:

$$
(5.27)\quad \begin{cases}
\text{(i)} & \|q^{(1)}(\cdot,t) - q^{(2)}(\cdot,t)\|_{L^1_\alpha(M)} \le C_1\|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)}\,\delta\gamma(M,t) \\
\text{(ii)} & \|q^{(1)}(\cdot,t) - q^{(2)}(\cdot,t)\|_{L^\infty_\alpha(M)} \le C\|h(\cdot,t)\|_{L^\infty_\alpha(M)}\,\delta\gamma(M,t) \\
\text{(iii)} & \|\frac{\partial q^{(1)}}{\partial t}(\cdot,t) - \frac{\partial q^{(2)}}{\partial t}(\cdot,t)\|_{L^1_\alpha(M)} \le C_1\|\frac{\partial\tilde{h}}{\partial x}(\cdot,t)\|_{L^1_\alpha(M)}\,\delta\gamma(M,t) \\
& + (C_2\delta\mu(M,t)\cdot\|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} + C_3\,\delta\eta(M,t) + C_4\,\delta\gamma(M,t)) \\
\text{(iv)} & \|\frac{\partial q^{(1)}}{\partial x}(\cdot,t) - \frac{\partial q^{(2)}}{\partial x}(\cdot,t)\|_{L^1_\alpha(M)} \le C_1\|\frac{\partial\tilde{h}}{\partial x}(\cdot,t)\|_{L^1_\alpha(M)}\,\delta\gamma(M,t) \\
& + C_5\|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)}\cdot\delta\gamma(M,t)
\end{cases}
$$

where C_j, $j = 1,\cdots,5$ depend only on $\gamma_\ell(M,t)$, $\eta_\ell(M,t)$, $\mu_\ell(M,t)$, $\lambda_\ell(M,t)$ $\quad \ell = 1,2$.

<u>Proof.</u> From (5.3), we have

$$
(5.28)\quad q^{(1)}(x,t) - q^{(2)}(x,t) = 2\int_0^\infty \tilde{h}(x+y,t)\{2(b^{(1)}(x,y,t) - b^{(2)}(x,y,t) + (b^{(1)} * b^{(1)} - b^{(2)} * b^{(2)})(x,y,t)\}dy
$$

whereupon

$$
(5.29)\quad \|q^{(1)}(\cdot,t) - q^{(2)}(\cdot,t)\|_{L^1_\alpha(M)} \le \int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)|\{2|b^{(1)} - b^{(2)}| + |b^{(4)} < b^{(4)} - b^{(2)} * b(2)|\}dydx
$$

$$
\le 2\|\tilde{h}(\cdot,t)_{L^1_\alpha(M)} \int_M^\infty e^{\gamma_1(x,t)}(1+\gamma_2(x,t)e^{\gamma_2(x,t)})\delta\eta(x,t)
$$

$$+ \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} \int_M^\infty (e^{\gamma_1(x,t)} \eta_1(x,t)$$

$$+ e^{\gamma_2(x,t)} \eta_2(x,t))e^{\gamma_1(x,t)} (1+\gamma_2(x,t)e^{\gamma_2(x,t)})\delta\eta(x,t)dx$$

$$\leq \|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} e^{\gamma_1(M,t)} (1+\gamma_2(M,t)e^{\gamma_2(M,t)})\delta\gamma(M,t)[2 +$$

$$(e^{\gamma_1(M,t)} \eta_1(M,t) + e^{\gamma_2(M,t)} \eta_2(M,t))].$$

The same idea for $L^\infty_\alpha(M)$ leads to (ii) above. Differentiating (5.28) with respect to t, we have:

$$\|\frac{\partial q^{(1)}}{\partial t}(\cdot,t) - \frac{\partial q^{(2)}}{\partial t}(\cdot,t)\|_{L^1_\alpha(M)} \leq (T) \cdot \|\frac{\partial \tilde{h}}{\partial t}(\cdot,t)\|_{L^1_\alpha(M)} \tag{5.30}$$

$$+ 2\int_M^\infty (1+|x|)^\alpha \int_0^\infty |\tilde{h}(x+y,t)|(|b_t^{(1)} - b_t^{(2)}|$$

$$+ |(b^{(1)} - b^{(2)}) * (b_t^{(1)} + b_t^{(2)})| + |(b_t^{(1)} - b_t^{(2)})$$

$$* (b^{(1)} + b^{(2)})|)dydx \leq \|\frac{\partial \tilde{h}}{\partial t}(\cdot,t)\|_{L^1_\alpha(M)} \cdot C + 2\|\tilde{h}(\cdot,t)\|_{L^1_\alpha(M)} \cdot D$$

where

$$D \equiv \int_M^\infty \{\|b_t^{(1)}(x,\cdot,t) - b_t^{(2)}(x,\cdot,t)\|_{L^\infty}(1+\|b^{(1)} + b^{(2)}(x,\cdot,t)\|_{L^1},$$

$$+ \|b^{(1)}(x,\cdot,t) - b^{(2)}(x,\cdot,t)\|_{L^\infty}\|b_t^{(1)} + b_t^{(2)}(x,\cdot,t)\|_{L^1}\}dx$$

$$\leq e^{\gamma_1(M,t)} (1+\gamma_2(M,t)e^{\gamma_2(M,t)})(1+\gamma_1(M,t)e^{\gamma_1(M,t)}$$

$$+ \gamma_2(M,t)e^{\gamma_2(M,t)})\{\delta\mu(M,t) + e^{\gamma_2(M,t)} \gamma_2(M,t)\delta\eta(M,t)\}$$

$$+ e^{\gamma_1(M,t,)} (1+\gamma_2(M,t)e^{\gamma_2(M,t)})\delta\gamma(M,t)$$

$$(e^{\gamma_1(M,t)}\mu_1(M,t)(1+\gamma_1(M,t)e^{\gamma_1(M,t)})$$

$$+\ e^{\gamma_2(M,t)}\mu_2(M,t)(1+\gamma_2(M,t)e^{\gamma_2(M,t)}).$$

Differentiating (5.28) with respect to x, we have:

$$\frac{\partial q^{(1)}}{\partial x}(x,t) - \frac{\partial q^{(2)}}{\partial x}(x,t) \equiv 2\int_0^\infty \frac{\partial \tilde{h}}{\partial x}(x+y,t)[b^{(1)} - b^{(2)}(x,y,t)]dy \tag{5.31}$$

$$+ \int_0^\infty \frac{\partial \tilde{h}}{\partial x}(x+y,t)[b^{(1)} * b^{(1)} - b^{(2)} * b^{(2)}]dy$$

$$+ \int_0^\infty \tilde{h}(x+y,t)\{2(b_x^{(1)} - b_x^{(2)}) + 2(b^{(1)} * b_x^{(1)} - b^{(2)} * b_x^{(2)})\}dy.$$

Estimating as in Lemma 5.4(iv), we obtain

$$\left\|\frac{\partial q^{(1)}}{\partial x}(\cdot,t) - \frac{\partial q^{(2)}}{\partial x}(\cdot,t)\right\|_{L^1_\alpha(M)} \leq C\left\|\frac{\partial \tilde{h}}{\partial x}(x,t)\right\|_{L^1_\alpha(M)} \tag{5.32}$$

$$+ 2\|\tilde{h}(\cdot,t)\|_{L^\infty(M)} \int_M^\infty dx \int_0^\infty dy(|b_x^{(1)} - b_x^{(2)}| + |b^{(1)} * b_x^{(1)} - b^{(2)} * b_x^{(2)}|)$$

$$\leq C\left\|\frac{\partial \tilde{h}}{\partial x}(\cdot,t)\right\|_{L^1_\alpha(M)} + 2\|\tilde{h}(\cdot,t)\|_{L^\infty_\alpha(M)} \int_M^\infty dx\{\|b_x^{(1)}(x,\cdot,t) -$$

$$b_x^{(2)}(x,\cdot,t)\|_{L^1}(1 + \|b^{(1)}_{(x,\cdot,t)} + b^{(2)}_{(x,\cdot,t)}\|_{L^1})$$

$$+ \|(b_x^{(1)} + b_x^{(2)})(x,\cdot,t)\|_{L^1}\|b^{(1)} - b^{(2)}(x,\cdot,t)\|_{L^1}\}$$

$$\leq C\left\|\frac{\partial \tilde{h}}{\partial x}(\cdot,t)\right\|_{L^1_\alpha(M)} + 2E\|\tilde{h}(\eta,t)\|_{L^\infty_\alpha(M)}\ \delta\gamma(M,t)$$

where

$$E \equiv \left(1 + \gamma_1 e^{\gamma_1} + \gamma_2 e^{\gamma_2}\right)\left\{\left(1+\gamma_1 e^{\gamma_1}\right)\left(1+\gamma_2 e^{\gamma_2}\right) + \gamma_2 e^{\gamma_2}\right\}$$

$$+ e^{\gamma_1}\left(1+\gamma_2 e^{\gamma_2}\right)(\gamma_1(1+\gamma_1) + \gamma_2(1+\gamma_2))$$

(for notational ease we have omitted the arguments of γ_1, γ_2 which are M,t). This follows from a straightforward estimate on $b_x^{(1)} - b_x^{(2)}$ and completes the proof of the lemma.

We are now ready to prove Theorem 5.1, as we have in fact derived Lipschitz estimates for the mapping $q \mapsto \Phi(q)$, whose Lipschitz constants depend linearly on various norms of $\tilde{h}(x,t)$.

Proof of Theorem 5.1. Let J be a compact subinterval of $\{t: 0 < t < \infty\}$. We define the Banach spaces B_M of functions via the norm

$$|||q|||_M \equiv \sup_{t\in J} \sum_{s+3r\leq 6} \{\|\partial_t^r \partial_x^s q(\cdot,t)\|_{L^1_{13/4-\delta-(3r+s)/2}(M)}$$

$$+ \|\partial_t^r \partial_x^s q(\cdot,t\|_{L^\infty_{4-(3r+s)/2}(M)}\}$$

where M will be suitably chosen, as discussed below.

By the estimates of the above lemmas and their direct extensions to the higher order derivatives contained in the norm above, the mapping $q \mapsto \Phi(q)$ maps B_M into itself for every M. Moreover, if we consider two functions q_1, q_2 in the unit ball of B_M (which therefore bounds $\sup_{t\in J} \gamma_j(M,t)$, $\eta_j(M,t)$, $\lambda_j(M,t)$, $\mu_j(M,t)$ by 1), then it is clear that an estimate of the form

$$|||\Phi(q_1) - \Phi(q_2)|||_M \leq K|||q_1 - q_2|||_M |||\tilde{h}|||_M$$

holds, where $K > 1$ is an absolute constant (involving the maximum of C,D,E as above). Thus choosing M so that $|||\tilde{h}|||_M < \frac{1}{K}$, Φ is a strict contraction and $\tilde{h}$ belongs to the unit ball in B_M. Hence on the interval $[M,\infty)$, equations (5.2), (5.3) have a unique solution $q(x,t)$.

To complete the proof of the Theorem 5.1, we must show that this solution extends to all finite x and that as $t \downarrow 0$, $q(x,t) \to Q(x)$ in $X^1_{3-\delta}$. The continuation argument follows that of [7] and proceeds as follows:

Suppose that a solution of (5.2), (5.3) in B_M exists only up to some finite value of M - i.e. M^*, the infimum of all M for which a solution exists for $x \geq M$ and belongs to B_M, is finite. Then for $\tilde{M} = M^* + \varepsilon$, we have a solution $q(\cdot,t) \in B_{\tilde{M}}$ for every $\varepsilon > 0$. There is a corresponding $b(x,y,t)$ via (5.2). Extend $b(x,y,t)$ for $x \geq M^* - \varepsilon$ as the constant (in x) function $b(\tilde{M},y,t)$. Since $\tilde{h}(x,t)$ exists for all x, we can consider (5.2), (5.3) on the x-interval $(M^* - \varepsilon, M^* + \varepsilon)$. By arguing as in the above lemmas, it is easy to show that our system has a unique solution for $b(x,y,t)$ in a ball relative to $b(\tilde{M},y,t)$ in a space with norm:

$$\sup_{\substack{x \in [M^*-\varepsilon, M^*+\varepsilon] \\ t \in J}} \sum_{\substack{3r+s+\sigma \leq 6 \\ r \leq 1}} \left\{ \| \partial_t^r \partial_x^s \partial_y^\sigma b(x,y,t) \|_{L^1_{(y)}} + \| \; \|_{L^\infty_{(y)}} \right\}$$

for ε sufficiently small. The corresponding $q(x,t)$ exists for $x \geq M^* - \varepsilon$ and belongs to $B_{M^*-\varepsilon}$, which contradicts the definition of M^*. Thus $M^* = -\infty$.

The proof that $q(x,t) \to Q(x)$ in $L^1_{3-\delta}(M)$ follows for M sufficiently large by iteration and for all semi-infinite intervals by continuation as above. This completes the proof of Theorem 5.1.

We remark that analogous results hold for initial data in L^1_σ with the same proofs. Note also that, as an immediate corollary of Theorem 5.1, we have the result that with L^1 initial data of compact support, the solution is C^∞ for $t > 0$ (see [6]).

6. Verification of the KdV Equation.

In order to prove that our generalized solution constructed above is indeed a solution of the KdV equation, it is sufficient, as remarked by Tanaka [19], to prove the following result

Theorem 6.1. $b_t(x,y,t) + b_{xxx}(x,y,t) - 3q(x,t)b_x(x,y,t) = 0.$

Proof. The idea is to show, using the Schrödinger equation and the trace formula, that $b_t + b_{xxx}$ and $3q\, b_x$ satisfy the same linear integral equation, which implies they are equal.

Differentiating the Volterra equation

$$b(x,y,t) = \int_{x+y}^{\infty} q(w,t)dw + \int_0^y \int_{x+y-z}^{\infty} q(w,t)b(w,z,t)dwdz \tag{6.1}$$

we obtain:

$$\begin{aligned} b_t(x,y,t) + b_{xxx}(x,y,t) &= \int_{x+y}^{\infty} (q_t(w,t) + q_{www}(w,t))dw \\ &+ \int_0^y \int_{x+y-z}^{\infty} [q(w,t)(b_t(w,z,t) + b_{www}(w,z,t)) \\ &\qquad + (q_t(w,t) + q_{www}(w,t))b(w,z,t) + 3(q_w(w,t)b_w(w,z,t))_w]dwdz. \end{aligned} \tag{6.2}$$

The trace formula

$$\begin{aligned} q(x,t) &= h(x,t) + 2\int_0^{\infty} h(x+y,t)b(x,y,t)dy \\ &+ \int_0^{\infty} h(x+y,t)\int_0^y b(x,y-z,t)b(x,z,t)dzdy \end{aligned} \tag{6.3}$$

when differentiated (recalling $h_t + h_{xxx} = 0$) yields

$$(6.4)\qquad q_t(x,t) + q_{xxx}(x,t) = 2\int_0^\infty H(x+y,t)(b_t(x,y,t) + b_{xxx}(x,y,t))dy$$

$$+ 2\int_0^\infty h(x+y,t)\cdot\int_0^y [b(x,y-z,t)(b_t(x,y,t) + b_{xxx}(x,y,t))$$

$$+ 3\, b_x(x,y-z,t)b_{xx}(x,y,t)]dzdy$$

$$+ 6\frac{\partial}{\partial x}\{\int_0^\infty h_x(x+y,t)[b_x(x,y,t) + \int_0^y b(x,y-z,t)b_x(x,z,t)dz]\cdot dy\}.$$

The estimates of Section 5 ensure the convergence of all these integrals. Combining (6.2), (6.4), we obtain the following integral equation for $\psi(x,y,t) \equiv b_t(x,y,t) + b_{xxx}(x,y,t)$:

$$\psi(x,y,t) = 2\int_{x+y}^\infty [(h * \psi)(w,t) + (h * (\psi * b))(w,t)$$

$$(6.5)$$

$$+ 3(h * (b_w * b_{ww}))(w,t) + 3(h_w * (b_w + b * b_w))_w(w,t)]dw$$

$$+ \int_0^y \int_{x+y-z}^\infty [q(w,t)\psi(w,z,t) + 3(q_w(w,t)b_w(w,z,t))_w$$

$$+ 2\, b(w,z,t)\{(h * \psi)(w,t) + (h * (\psi * b))(w,t)$$

$$+ 3(h * (b_w * b_{ww}))(w,t) + 3(h_w * (b_w + b * b_w))_w(w,t)\}]dwdz.$$

Here we use $*$ to denote both types of convolutions in (6.4) above. Thus it suffices to show that $3\, q(x,t)\cdot b_x(x,y,t)$ is also a solution of (6.5). Making this substitution and dividing by 3, we wish to show:

$$(6.6)\qquad q(x,t)b_x(x,y,t) \overset{?}{=} 2\int_{x+y}^{\infty} [(h * qb_w)(w,t) + (h * (qb_w * b))(w,t)$$

$$+ (h * (b_w * b_{ww}))(w,t) + (h_w * (b_w + b * b_w))_w(w,t)]dw$$

$$+ \int_0^y \int_{x+y-z}^{\infty} [q^2(w,t)b_w(w,z,t) + (q_w(w,t)b_w(w,z,t))_w$$

$$+ 2b(w,z,t)\{(h * qb_w)(w,t) + (h * (qb_w * b))(w,t)$$

$$+ (h * (b_w * b_{ww}))(w,t) + (h_w * (b_w + b * b_w))_w(w,t)\}]dwdz$$

From here on, the proof is a matter of direct calculation, using the following ingredients:

$$(6.7)\quad \begin{cases} \text{(i)} & \text{If } \phi(x,y,t) \in C^2,\ \phi \to 0 \text{ as } x \to_+ \infty, \text{ then} \\ & \phi(x,y,t) = \phi(x+y,0,t) + \int_0^y \int_{x+y-z}^{\infty} (\phi_{ww} - \phi_{wz})(w,z,t)dwdz \\ \text{(ii)} & b_{ww}(w,z,t) = q(w,t)b(w,z,t) + b_{wz}(w,z,t) \\ & \text{and } b_w(w,0,t) = b_z(w,0,t) = -q(w,t) \\ \text{(iii)} & \int_0^y u(z)v_y(y-z)dz = -u(y)v(0) + \frac{d}{dy}\{\int_0^y u(z)v(y-z)dz\}. \end{cases}$$

Using (6.7)(i) to re-express the left-hand side of (6.6) as

$$(6.8)\qquad -q^2(x+y,t) + \int_0^y \int_{x+y-z}^{\infty} [(q(w,t)b(w,z,t))_{ww} - (q(w,t)b(w,z,t))_{wz}]dwdz$$

and using (6.7)(ii) to eliminate b_{ww} terms, (6.6) is equivalent to:

$$(6.9)\qquad I(qq_w) \overset{?}{=} I(\rho)$$

where, for any function $g(w,t)$, we define

$$(6.10)\qquad I(g(w,t)) \equiv \int_{x+y}^{\infty} g(w,t)dw + \int_0^y \int_{x+y-z}^{\infty} g(w,t)b(w,z,t)dwdz,$$

and where $\rho(w,t)$ is given by

$$\text{(6.11)}\quad \rho(w,t) \equiv q(w,t)[(h * b_w)(w,t) + (h_w * b)(w,t) + 2(h * b * b_w)(w,t) + (h_w * b * b)(w,t)]$$
$$+ (h_{ww} * b)(w,t) + (h_w * (b_{wz})(w,t)$$
$$+ (h_{ww} * (b * b_w))(w,t) + (h_w * (b * b_{wz}))(w,t)$$
$$+ (h_w * b_w * b_w)(w,t) + (h * b_w * b_{wz})(w,t).$$

Clearly (6.9) holds if $qq_w = \rho$. Now, the terms in ρ which do not contain the factor q may be grouped pairwise. For the first pair, we have:

$$\text{(6.12)}\quad \int_0^\infty [h_{ww}(w+y,t)b_w(w,y,t) + h_w(w+y,t)b_{wy}(w,y,t)]dy$$
$$= h_w(w+y,t)b_w(w,y,t)\Big|_{y=0}^{y=\infty} = h_w(w,t)q(w,t).$$

The second pair is:

$$\text{(6.13)}\quad \int_0^\infty [h_{ww}(w+y,t) \cdot \int_0^y b(w,z,t)b_w(w,y-z,t)dz + h_w(w+y,t) \int_0^y b(w,z,t)b_{wy}(w,y-z,t)dz]dy$$
$$= \int_0^\infty (h_w(w+y,t) \int_0^y b(w,z,t)b_w(w,y-z,t)dz)_y dy$$
$$+ \int_0^\infty h_w(w+y,t)b(w,y,t)q(w,t)dy$$
$$= q(w,t) \int_0^\infty h_w(w+y,t)b(w,y,t)dy \quad \text{by (6.7)(iii).}$$

A similar reduction on the last pair of terms in ρ implies

$$\text{(6.14)}\quad \rho(w,t) = q(w,t)\ [\ h_w(w,t) + 2(h * b_w)(w,t) + 2(h_w * b)(w,t) + 2(h * b * b_w)(w,t) + (h_w * (b * b))(w,t)]$$

which by the trace formula (6.3) implies $\rho = q\, q_w$. Thus $b_t + b_{xxx} - 3\, qb_x = 0$, and our generalized solution of the KdV equation is indeed a bona fide solution.

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		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert L. Sachs		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-7927062,Mod.1
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18		12. REPORT DATE November 1981
		13. NUMBER OF PAGES 54
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

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18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

KdV equation, Cauchy problem, Inverse scattering

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Cauchy problem for the Korteweg-deVries equation (KdV for short)

$$(*)\quad \begin{cases} q_t(x,t) + q_{xxx}(x,t) - 6q(x,t)q_x(x,t) = 0 \\ q(x,0) = Q(x) \end{cases}$$

is solved classically under the single assumption (continued)

20. Abstract (continued)

$$\int_{-\infty}^{\infty} (1 + |x|^4)|\rho(x)|dx < \infty$$

for $t > 0$ via the so-called "inverse scattering method". This approach, originating with Gardner, Greene, Kruskal, and Miura [9], relates the KdV equation to the one-dimensional Schrödinger equation:

$$(**) \qquad -f''(x,k) + u(x)f(x,k) = k^2 f(x,k).$$

By considering the effect on the scattering data associated to the Schrödinger equation (**) when the potential $u(x)$ evolves in t according to the KdV equation (*), one obtains a linear evolution equation for the scattering data. The inverse scattering method of solving (*) consists of calculating the scattering data for the initial value $\rho(x)$, letting it evolve to time t, and then recovering $q(x,t)$ from the evolved scattering data.

Recently, P. Deift and E. Trubowitz [7] presented a new method for solving the inverse scattering problem (obtaining the potential from its scattering data). Our solution of the KdV initial value problem uses this approach to construct a classical solution under the assumption stated above.

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